JSc EUSL (2021), vol. 12, no. 2, p 47-60 *© 2021 Faculty of Science, Eastern University, Sri Lanka*

RESEARCH ARTICLE

AN ALTERNATIVE APPROACH FOR THE ANTI-MAGIC LABELLING OF A WHEEL GRAPH AND A PENDANT GRAPH

K. M. P. G. S. C. Kapuhennayake [1], A. A. I. Perera [1]*

[1]*Department of Mathematics, Faculty of Science, University of Peradeniya, Sri Lanka*

ABSTRACT

The Anti-magic labelling of a graph G with m edges and n vertices, is a bijection from the set of edges to the set of integers $\{1, \ldots, m\}$ such that all 'n' vertex summations are pairwise distinct. The vertex summation is the summation of the labels assigned to edges incident to a vertex. There is a conjecture that all simple connected graphs except K_2 are anti-magic. In our research, we found an alternative anti-magic labelling method for a wheel graph and a pendant graph. Wheel graph is a graph that contains a cycle of length $n-1$ and for which every graph vertex in the cycle is connected to one other graph vertex known as the "hub". The edges of a wheel, which connect to the hub are called "spokes". Pendant graph is a corona of the form $C_n \Theta K_1$ where $n \geq 3$. We label both wheel graph and pendant graph using the concept of the anti-magic labelling method of the path graph P_{n-1}. For wheel graph, we removed the middle vertex of the wheel graph and created a path graph using the vertices in the outer cycle of the wheel graph. Then the spokes of the wheel graph are represented by adding one edge to each vertex. For Pendant graph, we created a path graph using the cycle of the pendant graph and connect the pendant vertices to every vertex of the path graph. In both cases, we label all the edges using the concept of the anti-magic labelling of path graph $P_{n-1.}$ Finally, we calculated the vertex sum for each vertex and proved that every vertex sums are distinct and in the wheel graph, middle vertex takes the highest value.

Keywords: *Anti-magic labelling, wheel graph, pendant graph, path graph*

DOI. *http://doi.org/10.4038/jsc.v12i2.35*

1. INTRODUCTION

Anti-Magic labelling comes from its connection to magic labelling and magic squares. A magic square is a square array of numbers consisting of the distinct positive integers

*Corresponding author: *sonalichamathka03@gmail.com*

arranged such that the sum of numbers in any horizontal, vertical and main diagonal lines are always the same number. Likewise, motivated from the magic labelling, anti-magic then comes from being the opposite of magic. That is arranging numbers in a way such that the sums of numbers in the horizontal, vertical and main diagonal lines are distinct.

The concept of anti-magic labelling was introduced by Hartsfield and Ringel in 1989 (Hartsfield & Ringel, Pearls in graph theory, 1990). They defined it as follows: An Anti-Magic labelling of a graph G with m edges and n vertices, is a bijection from the set of edges to the set of integers $\{1, \ldots, m\}$ such that all 'n' vertex summations are pairwise distinct. The vertex sum is the summation of labels of all edges incident with that vertex. They conjectured that all simple connected graphs except K_2 are anti-magic (Hartsfield & Ringel, Pearls in graph theory, 1990).

There are several applications in anti-magic labelling. It could serve as model of surveillance or security system in civil engineering and circuit design, urban planning, electrical switchboards, communication networks.

In this paper, we introduce an alternative method for anti-magic labelling of wheel graph and pendant graph using the anti-magic labelling of path graph (Chang, Chen, & Li, 2021).

2. MATERIAL AND METHODS

First, we state some important definitions we use to get the result of this work.

Definition 2.1 (Anti-Magic labelling): Let G be a simple graph with V vertices and E edges. Anti-magic labelling of graph G is a one-to-one correspondence between $E(G)$ and $\{ 1,2, \ldots, |E| \}$ such that the vertex sum for distinct vertices is different (Sugeng, 2005). Vertex sum is the sum of the labels assigned to edges incident to a vertex.

Definition 2.2 (Wheel Graph): The wheel graph W_n of order n, is a graph that contains a cycle of length $n-1$ and for which every graph vertex in the cycle is connected to one other graph vertex known as the "hub". The edges of a wheel which connect to the hub are called "spokes" (Weisstein, Wheel Graph, n.d.).

Definition 2.3 (Corona of graphs G_1 and G_2): The corona of graphs G_1 and G_2 is the graph obtained by taking one copy of G_1, which has P_1 vertices and P_1 copies of G_2, and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 (Graf, 2014).

Definition 2.4 (Pendant graph): A pendant graph is a corona of the form $C_n \Theta K_1$ where $n \geq 3$ (Graf, 2014).

Definition 2.5 (Path graph): A path graph is a graph which has at least two connected vertices and has at least two terminal vertices (vertices that have degree 1), while all other (if any) have degree two (Weisstein, Path Graph, n.d.).

Since the path graphs are anti-magic we try to convert the wheel graph and the pendant graph into a path graph. (Chang, Chen, & Li, 2021).

First, consider the wheel graph. Represent the vertices in outer cycle as a path graph and then join the first vertex and last vertex by an edge. Spokes of the wheel graph are represented by adding one edge to each vertex.

Theorem 2.1: A path graph P_{n-1} is anti-magic and the wheel graph W_n constructed by path graph P_{n-1} is also anti-magic.

Proof: Represent the vertices in outer cycle as a path graph and join the first vertex and last vertex by an edge. Then spokes of the wheel graph can be represented by adding one edge to each vertex.

Let vertices on the outer cycle be $V_1 = \{(u_i, v_i)\ :\ i = 1,2, \dots , n-1\ \}$ and the middle vertex (hub) be V_2.

Define the edge label as follows:

Case 1: n is even

Edges in the outer cycle:

$$\vdots$$

$$f((u\,(\tfrac{n}{2}-3),v_1),(\,u\,(\tfrac{n}{2}-2),v_1)) = n-5;$$

$$f((u\,(\tfrac{n}{2}-2),v_1),(\,u\,(\tfrac{n}{2}-1),v_1)) = n-3;$$

$$f((u\,(\tfrac{n}{2}-1),v_1),(\,u\,(\tfrac{n}{2}),v_1)) = n-1;$$

$$f((u\,(\tfrac{n}{2}),v_1),(\,u\,(\tfrac{n}{2}+1),v_1)) = n-2;$$

$$f((u\,(\tfrac{n}{2}+1),v_1),(\,u\,(\tfrac{n}{2}+2),v_1)) = n-4;$$

$$\vdots$$

and Spokes:

$$\vdots$$

$$f((u(\tfrac{n}{2}-2),v_1),(v_2)) = 2n-5;$$

$$f\,((u(\tfrac{n}{2}-1),,v_1),(v_2)) = 2n-3;$$

$$f((u(\tfrac{n}{2}),,v_1),(v_2)) = 2n - 2;$$

$$f((u(\tfrac{n}{2}+1),,v_1),(v_2)) = 2n - 4;$$

$$f((u(\tfrac{n}{2}+2),,v_1),(v_2)) = 2n - 6;$$

$$\vdots$$

Edge labelling of wheel graph W_n (where n is even) can be represented by Figure 01.

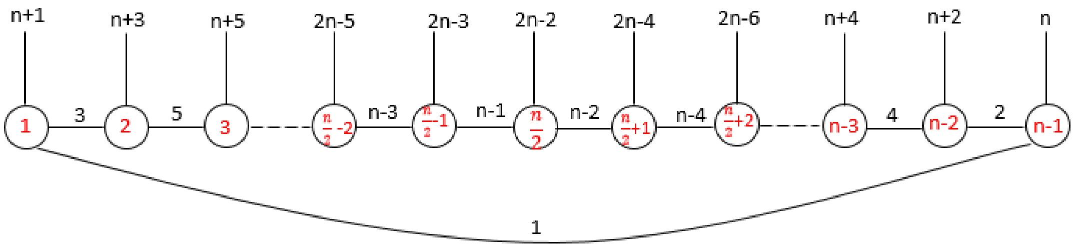

Figure 01: Edge labelling method for even n

Case 2: n is odd

Edges in the outer cycle:

$$\vdots$$

$$f((u\,(\tfrac{n-1}{2} - 3),v_1),(\,u\,(\tfrac{n-1}{2} - 2),v_1)) = n - 6;$$

$$f((u\,(\tfrac{n-1}{2} - 2),v_1),(\,u\,(\tfrac{n-1}{2} - 1),v_1)) = n - 4;$$

$$f((u\,(\tfrac{n-1}{2} - 1),v_1),(\,u\,(\tfrac{n-1}{2}),v_1)) = n - 2;$$

$$f((u\,(\tfrac{n-1}{2}),v_1),(\,u\,(\tfrac{n-1}{2} + 1),v_1)) = n - 1;$$

$$f((u\,(\tfrac{n-1}{2} + 1),v_1),(\,u\,(\tfrac{n-1}{2} + 2),v_1)) = n - 3;$$

$$f((u\,(\tfrac{n-1}{2} + 2),v_1),(\,u\,(\tfrac{n-1}{2} + 3),v_1)) = n - 5;$$

$$\vdots$$

and Spokes:

$$\vdots$$

$$f((u(\tfrac{n-1}{2} - 2),v_1),(v_2)) = 2n - 6;$$

$$f\,((u(\tfrac{n-1}{2} - 1),,v_1),(v_2)) = 2n - 4;$$

$$f((u(\tfrac{n-1}{2}),,v_1),(v_2)) = 2n-2;$$

$$f((u(\tfrac{n-1}{2}+1),,v_1),(v_2)) = 2n-3;$$

$$f((u(\tfrac{n-1}{2}+2),,v_1),(v_2)) = 2n-5;$$

$$\vdots$$

Edge labelling of wheel graph W_n (where n is odd) can be represented by figure 02.

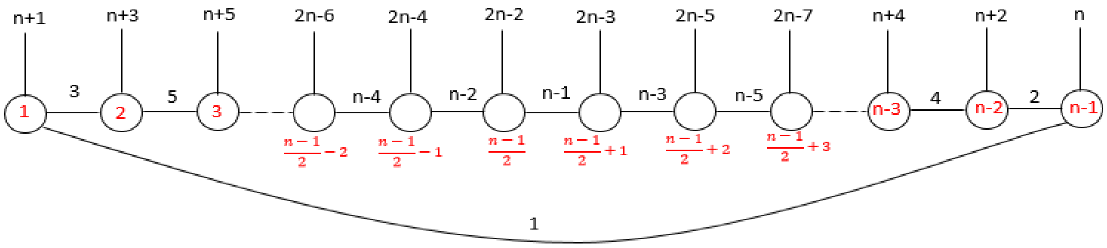

Figure 02: Edge labelling method for odd n

For a counter example, take $n = 6$. Edge labeling of wheel graph (W_6) can be represented by

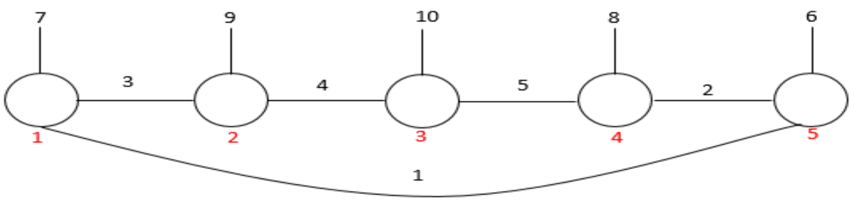

Figure 03: Edge labelling of W_6

Figure 03 shows the path graph constructed by the outer cycle of the wheel graph.

Figure 04 represents the final result of anti-magic labelling of Wheel graph with order 6.

Since the path graphs are anti-magic we try to convert the pendant graph into a path graph (Chang, Chen, & Li, 2021).

Now, let us represent the vertices in cycle graph as a path graph and join the first vertex and last vertex by an edge. Pendant vertices of the pendant graph are represented by adding one edge to each vertex.

Let vertices on the cycle graph be $V_1=\{(u_i,v_i): i = 1,2,\ldots,n\}$ and the middle vertex (hub) be $V_2 = \{(u_i,v_i): i = 1,2,\ldots,n\}$.

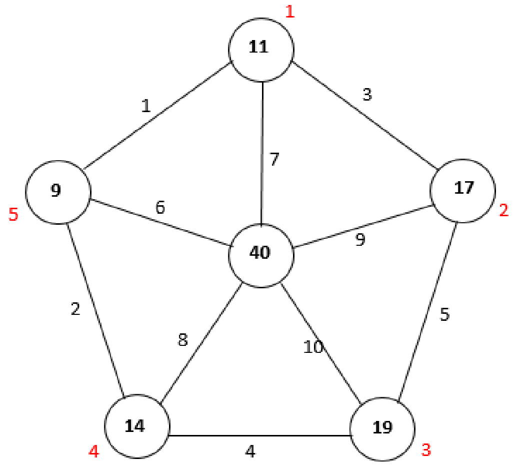

Figure 04: Anti-magic labelling of W_6

Theorem 2.2: A path graph P_{n-1} is anti-magic and the pendant graph W_n constructed by path graph P_{n-1} is also anti-magic.

Define the edge label as follows:

Case 1: *n* is even

Edges in the cycle graph:

$$\vdots$$

$$f((u(\tfrac{n}{2}-3),v_1),(u(\tfrac{n}{2}-2),v_1)) = \tfrac{n}{2}-3;$$

$$f((u(\tfrac{n}{2}-2),v_1),(u(\tfrac{n}{2}-1),v_1)) = \tfrac{n}{2}-2;$$

$$f((u(\tfrac{n}{2}-1),v_1),(u(\tfrac{n}{2}),v_1)) = \tfrac{n}{2}-1;$$

$$f((u(\tfrac{n}{2}),v_1),(u(\tfrac{n}{2}+1),v_1)) = \tfrac{n}{2};$$

$$f((u(\tfrac{n}{2}+1),v_1),(u(\tfrac{n}{2}+2),v_1)) = \tfrac{n}{2}+1;$$

$$\vdots$$

and Spokes,

$$\vdots$$

$$f((u(\tfrac{n}{2}-2),v_1),(v_2)) = \tfrac{3n}{2}+3;$$

$$f((u(\tfrac{n}{2}-1),,v_1),(v_2)) = \tfrac{3n}{2}+2;$$

$$f((u(\tfrac{n}{2}),,v_1),(v_2)) = \tfrac{3n}{2}+1;$$

$$f((u(\tfrac{n}{2}+1),,v_1),(v_2)) = \tfrac{3n}{2};$$

$$f((u(\tfrac{n}{2}+2),,v_1),(v_2)) = \tfrac{3n}{2}-1;$$

$$\vdots$$

Edge labelling of pendant graph $C_n \odot K_1$ (where n is even) can be represented by figure 05.

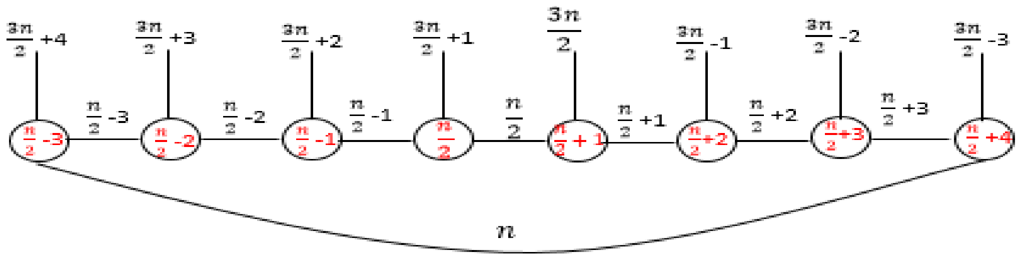

Figure 05: Edge labelling method for even n

Case 2: *n* is odd

Edges in the cycle graph:

$$\vdots$$

$$f((u(\tfrac{n+1}{2}-2),v_1),(u(\tfrac{n+1}{2}-1),v_1)) = \tfrac{n+1}{2}-2;$$

$$f((u(\tfrac{n+1}{2}-1),v_1),(u(\tfrac{n}{2}),v_1)) = \tfrac{n+1}{2}-1;$$

$$f((u(\tfrac{n+1}{2}),v_1),(u(\tfrac{n}{2}+1),v_1)) = \tfrac{n+1}{2};$$

$$f((u(\tfrac{n+1}{2}+1),v_1),(u(\tfrac{n}{2}+2),v_1)) = \tfrac{n+1}{2}+1;$$

$$f((u(\tfrac{n+1}{2}+2),v_1),(u(\tfrac{n}{2}+3),v_1)) = \tfrac{n+1}{2}+2;$$

$$\vdots$$

and Spokes:

$$\vdots$$

$$f((u(\tfrac{n+1}{2}-2),v_1),(v_2)) = \tfrac{3n+1}{2}+2$$

$$f\,((u(\tfrac{n+1}{2}-1),,v_1),(v_2)) = \tfrac{3n+1}{2}+1$$

$$f((u(\tfrac{n+1}{2}),,v_1),(v_2)) = \tfrac{3n+1}{2}$$

$$f((u(\tfrac{n+1}{2}+1),,v_1),(v_2)) = \tfrac{3n+1}{2}-1$$

$$f((u(\tfrac{n+1}{2}+2),,v_1),(v_2)) = \tfrac{3n+1}{2}-2$$

$$\vdots$$

Edge labelling of pendant graph $C_n \Theta K_1$ (where n is odd) can be represented by figure 06.

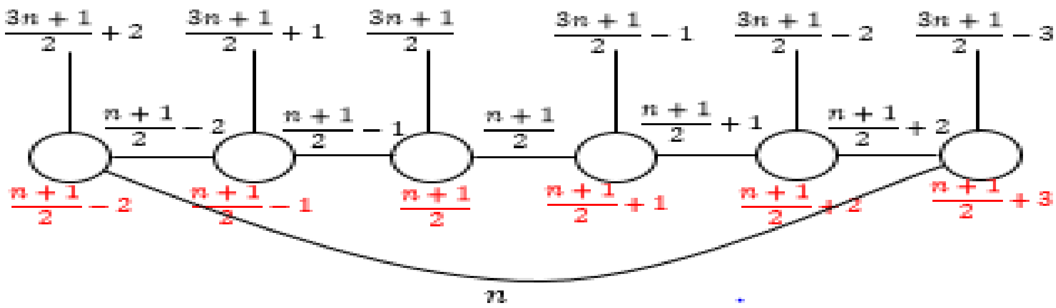

Figure 06: Edge labelling method for odd n

For a counter example, take $n = 5$. Edge labeling of pendant graph $(C_5 \Theta K_1)$ can be represented by Figure 07.

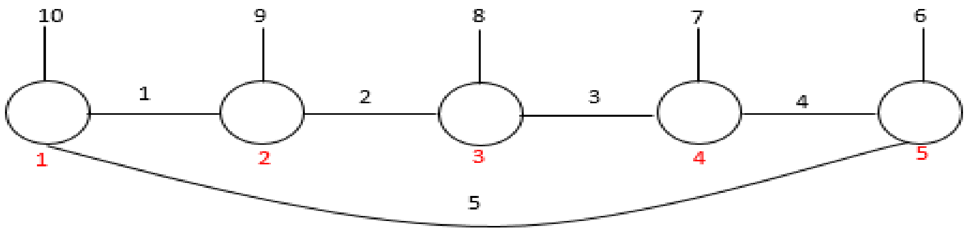

Figure 07: Edge labelling of pendant graph $C_5 \Theta K_1$

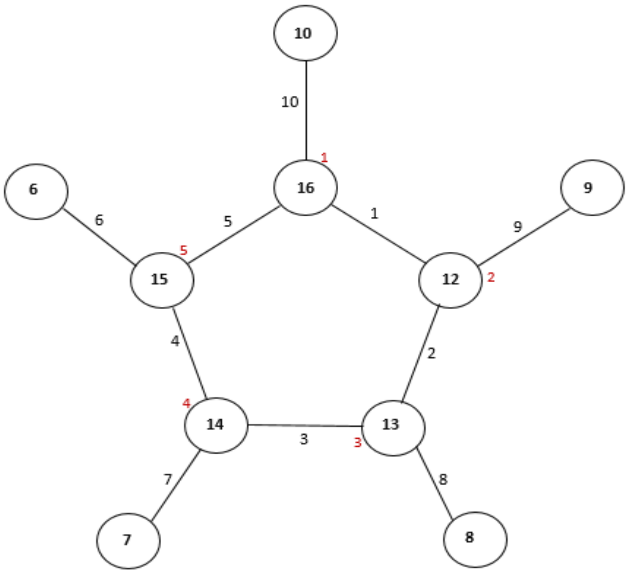

Figure 08: Anti-magic labelling of pendant graph $C_5 \Theta K_1$

3. RESULTS AND DISCUSSION

Consider the vertex summation of each edges of the wheel graph.

The general equation for vertex sums;

For even n

$$\left(\frac{n}{2} - m\right)^{\text{th}} \text{ vertex; } 4n - [7 + 6(m-1)]$$

$$\left(\frac{n}{2} - 1\right)^{\text{th}} \text{ vertex; } 4n - 7$$

$$\left(\frac{n}{2}\right)^{\text{th}} \text{ vertex; } 4n - 5$$

$$\left(\frac{n}{2} + 1\right)^{\text{th}} \text{ vertex; } 4n - 10$$

$$\left(\frac{n}{2} + m\right)^{\text{th}} \text{ vertex; } 4n - [10 + 6(m-1)];$$

where $n \in \mathbb{Z}^+$, $m = 2,3,4, \ldots$

For odd n

$$\left(\frac{n}{2} - m\right)^{\text{th}} \text{ vertex ; } 4n - [10 + 6(m-1)]$$

$$\left(\frac{n}{2} - 1\right)^{\text{th}} \text{ vertex; } 4n - 10$$

$$\left(\frac{n}{2}\right)^{\text{th}} \text{ vertex; } 4n - 5$$

$$\left(\frac{n}{2} + 1\right)^{\text{th}} \text{ vertex; } 4n - 7$$

$$\left(\frac{n}{2}+m\right)^{\text{th}} \text{ vertex; } 4n-[7+6(m-1)] \;;$$

where $n \in \mathbb{Z}^{+}$, $m = 2,3,4, \ldots$

Table 01: Vertex summation of outer circle

n even					n odd				
vertex	Left edge	Right edge	spoke	Vertex sum	vertex	Left vertex	Right vertex	spoke	Vertex sum
$\frac{n}{2}$-5	n-11	n-9	2n-11	4n-31	$\frac{n-1}{2}$-5	n-12	n-10	2n-12	4n-34
$\frac{n}{2}$-4	n-9	n-7	2n-9	4n-25	$\frac{n-1}{2}$-4	n-10	n-8	2n-10	4n-28
$\frac{n}{2}$-3	n-7	n-5	2n-7	4n-19	$\frac{n-1}{2}$-3	n-8	n-6	2n-8	4n-22
$\frac{n}{2}$-2	n-5	n-3	2n-5	4n-13	$\frac{n-1}{2}$-2	n-6	n-4	2n-6	4n-16
$\frac{n}{2}$-1	n-3	n-1	2n-3	4n-7	$\frac{n-1}{2}$-1	n-4	n-2	2n-4	4n-10
$\frac{n}{2}$	n-1	n-2	2n-2	4n-5	$\frac{n-1}{2}$	n-2	n-1	2n-2	4n-5
$\frac{n}{2}$+1	n-2	n-4	2n-4	4n-10	$\frac{n-1}{2}$+1	n-1	n-3	2n-3	4n-7
$\frac{n}{2}$+2	n-4	n-6	2n-6	4n-16	$\frac{n-1}{2}$+2	n-3	n-5	2n-5	4n-13
$\frac{n}{2}$+3	n-6	n-8	2n-8	4n-22	$\frac{n-1}{2}$+3	n-5	n-7	2n-7	4n-19
$\frac{n}{2}+4$	n-8	n-10	2n-10	4n-28	$\frac{n-1}{2}+4$	n-7	n-9	2n-9	4n-25

Proof:

Let us assume that any two vertex sums have same number.

(i) $4n-[7+6(m-1)] = 4n-7$

$$m = 1.$$

Contradiction since $m = 2,3,4, ..$

$$4n-[7+6(m-1)] = 4n-5$$
$$m = 2/3,$$

a contradiction since $m \in \mathbb{Z}^+$.

$$4n - [7 + 6(m-1)] = 4n - 10$$

$$m = 3/2,$$

a contradiction since $m \in \mathbb{Z}^+$.

(ii) $$4n - [10 + 6(m-1)] = 4n - 7$$

$$m = 1/2$$

a contradiction since $m \in \mathbb{Z}^+$.

(iii) $$4n - [10 + 6(m-1)] = 4n - 5$$

$$m = 1/6$$

a contradiction since $m \in \mathbb{Z}^+$.

(iv) $$4n - [10 + 6(m-1)] = 4n - 10$$

$$m = 1.$$

a contradiction. Since $m = 2,3,4,..$

(v) $$4n - [7 + 6(m-1)] = 4n - [10 + 6(m-1)]$$

$$7 = 10$$

a contradiction.

Therefore, no two vertex values are same. That is, every vertex sum is distinct. Hence, the wheel graphs W_n is anti-magic.

Now consider the vertex summation of each edges in pendant graph, which is shown in Table 2.

Now the general equation for vertex sums;

For odd n

$$\left(\tfrac{n+1}{2} - m\right)^{\text{th}} \text{ vertex} \quad ; \quad \tfrac{5n+3}{2} - (m+1)$$

$$\left(\tfrac{n+1}{2}\right)^{\text{th}} \text{ vertex} \quad ; \quad \tfrac{5n+3}{2} - 1$$

$$\left(\tfrac{n+1}{2} + m\right)^{\text{th}} \text{ vertex} \quad ; \quad \tfrac{5n+3}{2} - (m-1)$$

where $n, m \in \mathbb{Z}^+$.

Table 02: Vertex summation of pendant graph

n odd					n even				
Vertex	Left	Right	Pendant	Sum	Vertex	Left	Right	Pendant	Sum
$\frac{n+1}{2}-3$	$\frac{n+1}{2}-4$	$\frac{n+1}{2}-3$	$\frac{3n+1}{2}+3$	$\frac{5n+1}{2}-4$	$\frac{n}{2}-3$	$\frac{n}{2}-4$	$\frac{n}{2}-3$	$\frac{3n}{2}+4$	$\frac{5n}{2}-3$
$\frac{n+1}{2}-2$	$\frac{n+1}{2}-3$	$\frac{n+1}{2}-2$	$\frac{3n+1}{2}+2$	$\frac{5n+1}{2}-3$	$\frac{n}{2}-2$	$\frac{n}{2}-3$	$\frac{n}{2}-2$	$\frac{3n}{2}+3$	$\frac{5n}{2}-2$
$\frac{n+1}{2}-1$	$\frac{n+1}{2}-2$	$\frac{n+1}{2}-1$	$\frac{3n+1}{2}+1$	$\frac{5n+1}{2}-2$	$\frac{n}{2}-1$	$\frac{n}{2}-2$	$\frac{n}{2}-1$	$\frac{3n}{2}+2$	$\frac{5n}{2}-1$
$\frac{n+1}{2}$	$\frac{n+1}{2}-1$	$\frac{n+1}{2}$	$\frac{3n+1}{2}$	$\frac{5n+1}{2}-1$	$\frac{n}{2}$	$\frac{n}{2}-1$	$\frac{n}{2}$	$\frac{3n}{2}+1$	$\frac{5n}{2}$
$\frac{n+1}{2}+1$	$\frac{n+1}{2}$	$\frac{n+1}{2}+1$	$\frac{3n+1}{2}-1$	$\frac{5n+1}{2}$	$\frac{n}{2}+1$	$\frac{n}{2}$	$\frac{n}{2}+1$	$\frac{3n}{2}$	$\frac{5n}{2}+1$
$\frac{n+1}{2}+2$	$\frac{n+1}{2}+1$	$\frac{n+1}{2}+2$	$\frac{3n+1}{2}-2$	$\frac{5n+1}{2}+1$	$\frac{n}{2}+2$	$\frac{n}{2}+1$	$\frac{n}{2}+2$	$\frac{3n}{2}-1$	$\frac{5n}{2}+2$
$\frac{n+1}{2}+3$	$\frac{n+1}{2}+2$	$\frac{n+1}{2}+3$	$\frac{3n+1}{2}-3$	$\frac{5n+1}{2}+2$	$\frac{n}{2}+3$	$\frac{n}{2}+2$	$\frac{n}{2}+3$	$\frac{3n}{2}-2$	$\frac{5n}{2}+3$
$\frac{n+1}{2}+4$	$\frac{n+1}{2}+3$	$\frac{n+1}{2}+4$	$\frac{3n+1}{2}-4$	$\frac{5n+1}{2}+3$	$\frac{n}{2}+4$	$\frac{n}{2}+3$	$\frac{n}{2}+4$	$\frac{3n}{2}-3$	$\frac{5n}{2}+4$

For even n

$$\left(\frac{n}{2}-m\right)^{\text{th}} \text{ vertex } \quad ; \frac{5n}{2}-m$$

$$\left(\frac{n}{2}\right)^{\text{th}} \text{ vertex } \quad ; \frac{5n}{2}$$

$$\left(\frac{n}{2}+m\right)^{\text{th}} \text{ vertex } \quad ; \frac{5n}{2}+m$$

Where $n, m \in \mathbb{Z}^+$.

Proof: Let us assume that any two vertex sums have same number.

For odd n

(i) $\frac{5n+3}{2}-1=\frac{5n+3}{2}-(m+1)$

$m=0,$

a contradiction since $m \in \mathbb{Z}^+$;

(ii) $\frac{5n+3}{2}-1=\frac{5n+3}{2}-(m-1)$

$m=0,$

a contradiction since $m \in \mathbb{Z}^+$;

(iii) $\frac{5n+3}{2}-(m+1)=\frac{5n+3}{2}+(m-1)$

$m=0,$

a contradiction since $m \in \mathbb{Z}^+$.

For even *n*

(iv) $\frac{5n}{2} = \frac{5n}{2} - m$

$m = 0,$

a contradiction since $m \in \mathbb{Z}^+$;

(v) $\frac{5n}{2} = \frac{5n}{2} + m$

$m = 0,$

a contradiction since $m \in \mathbb{Z}^+$;

(vi) $\frac{5n}{2} - m = \frac{5n}{2} + m$

$m = 0,$

a contradiction since $m \in \mathbb{Z}^+$.

Therefore, no two vertex values are same. That is, each vertex sum is distinct. Hence, the pendant graph is anti-magic.

CONCLUSION

In our work, we found an alternative method for anti-magic labelling of wheel graph and pendant graph using the anti-magic labelling of path graph. For a wheel graph, we removed the middle vertex of the wheel graph and created a path graph using the vertices in the outer cycle of the wheel graph. Then spokes of the Wheel graph are represented by adding one edge to each vertex. Using the anti-magic labelling method of the path graph P_{n-1}, we found an alternative method to label the edges of the outer cycle of the wheel graph. Finally found the vertex sum for each vertex and we proved that every vertex sum is distinct and the middle vertex take the highest value. Anti-magic labelling method for a pendant graph was found by creating a path graph using the cycle of the pendant graph and connecting the pendant vertices to every vertex of the path graph. For anti-magic labelling of both pendant graph and wheel graph, we used the concept of the anti-magic labelling of path graph $P_{n-1.}$ Finally, the vertex sum for each vertex was calculated and proved that every vertex sums take different values and the middle vertex take the highest value for the wheel graph.

REFERENCES

1. Chang, F. H., Chen, H. B., & Li, W. T. (2021). Shifted-Antimagic Labelings for Graphs. *Graphs and Combinatorics*, 1065–1082.

2. Graf, A. (2014). Graceful Labelings of Pendant Graphs . *Rose-Hulman Undergraduate Mathematics Journal*, 158-172.

3. Hartsfield, N., & Ringel, G. (1994). *Pearls in graph theory.* Boston: Academic press, INC, Boston.

4. Sugeng, K. A. (2005). *Magic and Antimagic Labeling of Graphs.* Victoria , Australia.

5. Weisstein, E. W. (n.d.). *Path Graph*. Retrieved from MathWorld--A Wolfram Web Resource: https://mathworld.wolfram.com/PathGraph.html

6. Weisstein, E. W. (n.d.). *Wheel Graph*. Retrieved from MathWorld--A Wolfram Web Resource: https://mathworld.wolfram.com/WheelGraph.html