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RESEARCH ARTICLE

A DEDUCTIBLE COMPARISON OF COST PER-LOSS, COST PER-PAYMENT EVENT, LOSS ELIMINATION RATIO UNDER EXPONENTIALLY AND LOGNORMALLY DISTRIBUTED SEVERITIES

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ABSTRACT

The maximum amount of losses retained by the insured under deductible policy modifications is usually set as part of the terms of the policy conditions. The objective of this study are to (i) estimate mean losses of an insured risk by means of the operational behavior of density function with deductible modifications and (ii) compare the mean severities under exponentially and log-normally distributed arbitrary policy in a cost per loss and cost per payment contingencies. The results show that despite the established fact in literature that log-normal severity distribution has a thicker tail than the exponential distribution, its cost per loss payment is correspondingly lower than the corresponding values of the exponential mean loss. Computational evidence over the trend of the change in the loss eliminated in the domain for which deductible has been defined revealed that the cost per loss amount is less than the cost per payment amount in the two models. The method can be used to estimate aggregate claims as the deductible level increases for every scheme holder and such that the estimated claims could be compared with the hypothetical observed claims which can be arrived at by applying the hypothetical deductible value to the background losses.

Keywords: *Cost per loss; Cost per payment; Deductible; Exponential distribution; Log-normal distribution*

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1. INTRODUCTION TO SINGULARITY FUNCTIONS

The objective of this study are to (i) estimate mean losses of an insured risk by means of the operational behavior of density function with deductible modifications and (ii) compare the mean severities under exponentially and log-normally distributed arbitrary policy in a cost per loss and cost per payment contingencies. Underwriters enforce deductibles as part of their underwriting processes in order to share risks with the policies they manage.

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By introducing deductibles and sharing the risk, the underwriter expects the insured to be more cautious. Deductible is a form of retention that provides economic incentives to the scheme holder to manage his own losses. In practice, the more an insured is actively participating in the cost of his claim, the better it will be to prevent nuisance claims. Consequently, smaller claims will be advised to the underwriter. Processing of claims such as verification costs could be significant for the insurance firm and this is the reason why underwriters will prefer to reduce the number of claims with lower amounts. However, the introduction of deductible amount may upset the insured because the losses will not be indemnified in full. We observe in Pacakova & Brebera (2015) and Zacaj et al. (2015) that the severity distributions used for risk theoretical analysis could be evaluated only after rigorous data processing because the generation of loss distribution arising from insurance data is very difficult. However deductible statistics is quite limited since database containing information on deductible, policy and claim may be missing or unavailaible. In Zacaj et al. (2015); Bakar et al. (2015), We also observe that claim generating processes is essentially tedious under social-economic conditions and consequently, the magnitude of claim could be obtained by the claim size management of an insurance contract. Following Raschke (2019), medium size claim could be subject of log-normal framework influenced by base distribution function $F_b(x)$ and base survival distribution $S_b(x)$ and resulting in a random risk. Insurance managers place strong emphasis on severities in relation to selecting an adequate probability model to analyse claims data. A clear knowledge of loss distribution in risk theory is therefore needed to summarise and model claims data. In actuarial discipline, the ability to understand and interpret claim data is a requirement to build a good claim model which allows us to make a critical underwriting judgement in estimating premiums, expected profit and reserve, hence the knowledge of the distribution of insurance claims data can further be used to advise underwriters on reinsurance decisions. In general insurance business, the objective of risk theory is to quantify and analyse the dimensions of the risk of severity. We infer from Tse (2009) that actuarial risk theory is responsible for building models of pricing based on observations of the random variables for the expected size of claim outgo $E\left(C(e)\right)=\sum_{i=1}^{\kappa(e)}h(x_i)E\left(I_{\kappa(e)>0}\right)=\sum_{i=1}^{\kappa(e)}h(x_i)\Pr(\kappa>0)$ and frequency of claims $\kappa(e)$ where $h(x_i)$ are function of random risk, the exposure e is expressed as $e=S\times l(\zeta)$, S is the sum insured of insurance under cover, $l(\zeta)$ is length of time during which period the scheme holder has been exposed to severity risk and $I_{\kappa(e)>0}$ is the indicator function. The main responsibility of a claim actuary in policy underwriting is to obtain appropriate value on the cash flows from the insurance industry. From the knowledge of the cash flows, actuarial models are constructed to describe and analyse cash flow process. General insurance including casualty and health insurance describe the most critically challenging sector for claims actuaries because it is driven by data where the cash outflow is the claim outgo. Analysis of the severity claim based models such as log-normal and exponential distribution using appropriate density functions form the basis of solving actuarial claim issues arising in general insurance business. In Afify et al (2020), claim managers seldom bother on occurrence of claim but are concerned with the random processes describing the severity value the claim managers pays as indemnity rather than the particular events resulting in the claims process. Consequently, claims manager should have a good knowledge of loss distribution models which consists of total amounts of claims payable by insurers over a defined period. Insurance industry is data driven probably in large amounts which could be infrequent and consequently, it is necessary to identify suitable density models having the characteristics of heavy tails such as log-normal distribution and high skewness such as exponential distribution.

Following Afify et al (2020), loss distribution is a description of risk exposure units e the degree of which is computed from risk indicator metrics which are functions of the model. Underwriting managers usually employ the risk metrics to assess the level at which the insurance firms are exposed to defined areas of risks arising from vagaries of underlying variables such as prices and interest rates. In Tse (2009), the upper tail of a severity size distribution in general insurance business can be modelled by log-normal distribution although the upper tail may not necessarily be log-normally distributed. The model should be adequate to the extent of enabling decisions relating to solvency requirements, loading, premium analysis, technical reserve, expected profit forecast, reinsurance and the influence of deductibles on severities. The size of a claim at a material time is of particular significance to the underwriting management of an insurance firm. The conditions under which claims data are obtained and future claims subsequently estimated is an enabling factor to estimate severity amounts in general insurance business as samples from definite but usually heavy tailed probability distributions. In view of Tse (2009), it is observed that as a probability based actuarial model for severity analysis, the probability of financial losses experienced by scheme holders and indemnified by insurance firm should be clearly understood under the contract setting.

Analytical actuarial distributions are deployed to assess the cost to the extent that such distributions are positively skewed having high probability densities on the right tails. Since the distribution are specifically applied to analyse losses, they are then tagged loss distribution models. Claim modelling remains the basis of information content for underwriting firms in order to obtain estimate of premium, loading, reserves, profits and capital required to ascertain overall profitability and to assess the effect of deductible regimes. Although it is reasonable to fit probability distributions to claim data, analytical probability distribution is rather powerful technique to employ in analyzing claims data and consequently, there is need to construct model which can be used to estimate the distribution of claims under exponentially and log-normally distributed actuarial data involving deductible clauses. For the purpose of this work, we are concerned with the analysis of log-normal and exponential distributions of claims estimations for policies having deductible clauses. For positive claims data, heavy tailed distributions are useful tools of analysis. In general insurance business, claims data are usually skewed to the right tail and any distribution showing this type of behavior is sufficient for the analysis of severities (Sakthivel & Rajitha, 2017). The choice of theses distributions is based on this principle of claims data to compare a light tailed and heavy tailed in the evaluation of deductible. Moreover, because of the small size of the data both the exponential and log-normal models can cover better the behavior of such losses (Ahmad, Mahmoudi, Hamedani & Kharazmi, 2020). An important characteristic of a probability distribution to meet the requirements of an heavy tailed probability density is that the distributions is expected to have a tail probabilities exceeding 1 that is $\lim_{y\to\infty}\left\{\frac{e^{-\theta y}}{S_Y(y)}\right\}=0$ for every $\theta > 0$, where $S_Y(y)$ is the survival function. It is assumed provided that there are no points of truncation in the data, the first moment of the distribution should exist. Claim modelling is therefore necessary because the construction of adequate explainable loss model serves as the foundation of critical underwriting decisions taken in relation to premiums and claims assessment to ascertain profitability.

2.0 ACTURIAL PRELIMINARIES OF DEDUCTIBLE

Deductibles is a fundamental concept in general insurance underwriting because it determines the frequency of times the scheme holder advises claim and it could also determine the value of indemnity payable to the policy holder in the event of a contingency. However, deductible could influence insurance claims to be observed with censoring and truncation which are usually dealt with during premium pricing of insurance contracts. Suppose κ represents the frequency of loss advised and X_i denotes the severity of the insurance loss which are assumed independent of one another. Furthermore, suppose we apply the retention D such that the risk function defined by

$$R(X_i, D) = \begin{cases} 0 & X_i < D \\ X_i & D \le X_i < \infty \end{cases} \tag{1}$$

Following Brown & Lennox (2015) and the definition of censoring and truncation in actuarial literature, we define the followings. The censored severity is

$$\kappa_R^i(D) = \begin{cases} 0 & X_i < D \\ X_i & D \le X_i < \infty \end{cases} \tag{2}$$

The truncated severity is

$$X_{\bullet,i}(D) = X_i - D \mid D \le X_i \tag{3}$$

Let

$$I(D < X_i) = \begin{cases} 1 & D < X_i \\ 0 & otherwise \end{cases} \tag{4}$$

$E\left(I\left(D<X_i\right)\right)=1\times\Pr\left(D<X_i\right)+0\times\Pr\left(D\geq X_i\right)=\Pr\left(D<X_i\right)$ be an indicator function then $\kappa_R\left(D\right)=\sum_{i=1}^{\kappa}I\left(D<X_i\right)$ is the number of claims.

The sum of the censored severity over i is,

$$\sum_{i=1}^{\kappa}\kappa_R^i\left(D\right)=\Sigma_R\left(D\right) \tag{5}$$

is the total claim. Following Brown and Lennox (2015), the deductible relativity ρ for a single loss on a single claim of an insurance contract is given by

$$Deductible\,relativity\left(\rho\right)=\frac{E\left(X_{\bullet,i}\left(D\right)\right)}{E\left(X\right)} \tag{6}$$

Consequently,

$$\frac{E\left(X_{\bullet,i}\left(D\right)\right)}{E\left(X\right)}=1-LER\left(X\right) \tag{7}$$

the loss elimination ratio

$$LER\left(y\right)=\frac{\langle X\rangle-E\left(X_R\left(D\right)\right)}{\langle X\rangle} \tag{8}$$

$$\frac{E\left(X_{\bullet,i}\left(D\right)\right)}{E\left(X\right)}=1-\frac{\langle X\rangle-E\left(X_R\left(D\right)\right)}{\langle X\rangle} \tag{9}$$

Following Ogungbenle (2020; 2021b)

$$E\left(X_R\left(D\right)\right)=\int_0^{\infty}xf_{X_R}\left(x\right)dx \tag{10}$$

$$E\left(X_R\left(D\right)\right)=\int_D^{\infty}\left(x-D\right)f_X\left(x\right)dx \tag{11}$$

Using $F_X\left(x\right)=1-S_X\left(x\right)\Rightarrow F_X'\left(x\right)=-S_X'\left(x\right)\Rightarrow f_X\left(x\right)=-S_X'\left(x\right)$

$$E\left(X_R\left(D\right)\right)=-\int_D^{\infty}\left(x-D\right)dS_X\left(x\right)=-\int_D^{\infty}\left(x-D\right)d\left(1-F_X\left(x\right)\right) \tag{12}$$

$$E\left(X_R\left(D\right)\right)=-\left\{\left[S_X\left(x\right)\left(x-D\right)\right]_D^{\infty}-\int_D^{\infty}\left(1-F_X\left(x\right)\right)dx\right\}=\int_D^{\infty}S_X\left(x\right)dx \tag{13}$$

$$\langle X\rangle=\int_0^{\infty}xf_X\left(x\right)dx \tag{14}$$

Theorem 2.1: Let

$$E\left(X_R\left(D\right)\right)=E\left(x-D\right).I\left(x>D\right) \tag{15}$$

If β is a random variable and I is indicator function. Then

(i) $$\frac{d^2 E\left(X_R\left(D\right)\right)}{dD^2} = f_X\left(D\right); \tag{16}$$

(ii) $$\frac{d}{dD}E\left(X_R\left(D\right)\right) + \int_0^\infty f_X\left(\beta\right)d\beta = \int_{-\infty}^{D} f_X\left(\beta\right)d\beta. \tag{17}$$

Proof:

We recognize that $\int_0^\infty f_X\left(\beta\right)d\beta = 1$ and we can express $E\left(X_R\left(D\right)\right)$ above as

$$E\left(X_R\left(D\right)\right) = \int_D^\infty \left(x-D\right) f_X\left(x\right)dx = E\left(x-D\right).I\left(x>D\right) = \int_D^\infty x f_X\left(x\right)dx - D\left(1-F_X\left(D\right)\right) \tag{18}$$

$$\frac{d}{dD}E\left(X_R\left(D\right)\right) = \frac{d}{dD}\left(\int_D^\infty x f_X\left(x\right)dx - D\left(1-F_X\left(D\right)\right)\right); \tag{19}$$

$$\frac{d}{dD}E\left(X_R\left(D\right)\right) = \frac{d}{dD}\left(\int_D^\infty x f_X\left(x\right)dx\right) + \frac{d}{dD}\left(-D\left(1-F_X\left(D\right)\right)\right). \tag{20}$$

Recall that if $\eta\left(s\right) = \int_{a(s)}^{b(s)} g\left(y,s\right)dy$ then

$$\frac{d\eta\left(s\right)}{ds} = \int_{a(s)}^{b(s)} \frac{\partial}{\partial s} g\left(y,s\right)dy + g\left(b\left(s\right),s\right)\frac{\partial}{\partial s}b\left(s\right) - g\left(a\left(s\right),s\right)\frac{\partial}{\partial s}a\left(s\right) \tag{20a}$$

Differentiating equation (20) with respect to D while applying (20a) to the integral, we have

$$\frac{d}{dD}E\left(X_R\left(D\right)\right) = \left(0 - D\times f_X\left(D\right)\frac{dD}{dD} + \int_D^\infty \frac{\partial}{\partial D}\left[x f_X\left(x\right)\right]dx\right) + \frac{d}{dD}\left(\left\{-D + D\times F_X\left(D\right)\right\}\right) \tag{21}$$

$$\frac{d}{dD}E\left(X_R\left(D\right)\right) = -D\times f_X\left(D\right) - 1 + F_X\left(D\right) + D\times F_X'\left(D\right). \tag{22}$$

Plugging $F_X'\left(D\right) = f_X\left(D\right)$ in (22), we have

$$\frac{d}{dD}E\left(X_R\left(D\right)\right) = -D\times f_X\left(D\right) - 1 + F_X\left(D\right) + D\times f_X\left(D\right); \tag{23}$$

$$\frac{d}{dD}E\left(X_R\left(D\right)\right) = F_X\left(D\right) - 1, \tag{24}$$

and consequently, equation (24) implies

$$\frac{d}{dD}E\left(X_R\left(D\right)\right) = \int_{-\infty}^{D} f_X\left(\beta\right)d\beta - 1. \tag{25}$$

Since $\int_{0}^{\infty} f_X(\beta)d\beta = 1$,

$$\frac{d}{dD}E\left(X_R(D)\right)+\int_{0}^{\infty} f_X(\beta)d\beta = \int_{-\infty}^{D} f_X(\beta)d\beta .$$

Differentiating both sides of equation (24) with respect to D again, we have

$$\frac{d^2E\left(X_R(D)\right)}{dD^2}=\frac{d}{dD}F_X(D)=F_X'(D); \tag{26}$$

$$\frac{d^2E\left(X_R(D)\right)}{dD^2}=f_X(D). \tag{27}$$

The essence of deductible relativity ρ is to investigate by how much the size of an insurance loss has been minimized by the deductible D from a per loss perspective. However, the function $LER(y)$ is a measure of how much the covered loss is minimized through the application of D. Suppose the insurance scheme has an upper limit of cover θ, the function

$$LER(y)=\frac{\int_{0}^{D} xf_X(x)dx+\int_{D}^{\theta} f_X(x)dx}{\int_{0}^{\theta} xf_X(x)dx}. \tag{28}$$

Following Brown & Lennox (2015), we let D_B denote the base deductible, then the relativity ρ of aggregate claim is given as follows

$$\rho(D_B,D)=\frac{E\left(\sum_{i=1}^{\kappa}\kappa_R^i(D)\right)}{E\left(\sum_{i=1}^{\kappa}\kappa_R^i(D_B)\right)}=\frac{\left(\sum_{i=1}^{\kappa}E\left(\kappa_R^i(D)\right)\right)}{\left(\sum_{i=1}^{\kappa}E\left(\kappa_R^i(D_B)\right)\right)}=\frac{E(\kappa)}{E(\kappa)}=1; \tag{29}$$

$$\rho(D_B,D)=\frac{E(\kappa)\int_{D}^{\theta} S_X(x)dx}{E(\kappa)\int_{D_B}^{\theta} S_X(x)dx}=\frac{\int_{D}^{\theta} S_X(x)dx}{\int_{D_B}^{\theta} S_X(x)dx}; \tag{30}$$

$$E\left(\Sigma_R(D_B)-\Sigma_R(D)\right)=E(\kappa)\left(E\left(X_R(D_B)\right)-E\left(X_R(D)\right)\right)= \\ E(\kappa)\left(\int_{D_B}^{D} S_X(x)dx\right)=E(\kappa)\left(\int_{D}^{\infty} S_X(x)dx-\int_{D_B}^{\infty} S_X(x)dx\right) \tag{31}$$

$$E\left(\Sigma_R(D_B)\right)=E\left(\Sigma_R(D)\right)+E(\kappa)\left(\int_{D}^{\infty} S_X(x)dx-\int_{D_B}^{\infty} S_X(x)dx\right); \tag{32}$$

$$E\left(\Sigma_R\left(D_B\right)\right)=E\left(\Sigma_R\left(D\right)\right)+E\left(\kappa\right)\left(\left\langle X_R\right\rangle_D-\left\langle X_R\right\rangle_{D_B}\right). \tag{33}$$

Theorem 2.2: Let ζ be a random loss with $E(\zeta)$. If

$$E\left(X_R\left(D\right)\right)=\int_D^\infty\left(x-D\right)d\left(1-S_X\left(x\right)\right). \tag{34}$$

Then

$$\lim_{D_a\to D_b}\left[\frac{1}{D_a-D_b}\int_{D_a}^{D_b}S_X\left(x\right)dx\right]=\frac{1}{E\left(\kappa\right)}\frac{d}{dD}E\left(X_R\left(D\right)\right). \tag{35}$$

Proof: From equation (12),

$$E\left(X_R\left(D\right)\right)=\int_D^\infty\left(x-D\right)d\left(1-S_X\left(x\right)\right). \tag{36}$$

Integrating (36) by parts, we have

$$E\left(X_R\left(D\right)\right)=\left[\left(x-D\right)\left(1-F_X\left(x\right)\right)\right]_D^\infty+\int_D^\infty\left(1-F_X\left(x\right)\right)dx=\int_D^\infty S_X\left(x\right)dx. \tag{37}$$

Similarly, $E\left(X_R\left(\overline{D}\right)\right)=\int_{\overline{D}}^\infty S_X\left(x\right)dx$ and consequently,

$$E\left(X_R\left(D\right)\right)-E\left(X_R\left(\overline{D}\right)\right)=\int_D^{\overline{D}}S_X\left(x\right)dx. \tag{38}$$

Following Brown & Lennox (2015),

$$E\left(X_R\left(D\right)\right)=E\left(\zeta\right)-E\left(\kappa\right)\int_0^D S_X\left(x\right)dx. \tag{39}$$

Substituting the deductibles D_a and D_b into equation (39) respectively, we have

$$E\left(X_R\left(D_a\right)\right)=E\left(\zeta\right)-E\left(\kappa\right)\int_0^{D_a}S_X\left(x\right)dx; \tag{40}$$

$$E\left(X_R\left(D_b\right)\right)=E\left(\zeta\right)-E\left(\kappa\right)\int_0^{D_b}S_X\left(x\right)dx. \tag{41}$$

Subtracting equation (41) from (40), We have

$$E\left(X_R\left(D_a\right)\right)-E\left(X_R\left(D_b\right)\right)=E\left(\zeta\right)-E\left(\kappa\right)\int_0^{D_a}S_X\left(x\right)dx-\left\{E\left(\zeta\right)-E\left(\kappa\right)\int_0^{D_b}S_X\left(x\right)dx\right\} \tag{42}$$

Simplifying the difference in (42), We have

$$E\left(X_R\left(D_a\right)\right)-E\left(X_R\left(D_b\right)\right)=E\left(\zeta\right)-E\left(\kappa\right)\int_0^{D_a}S_X\left(x\right)dx-E\left(\zeta\right)+E\left(\kappa\right)\int_0^{D_b}S_X\left(x\right)dx \tag{43}$$

$$E\left(X_R\left(D_a\right)\right)-E\left(X_R\left(D_b\right)\right)=E(\kappa)\int_0^{D_b} S_X(x)dx-E(\kappa)\int_0^{D_a} S_X(x)dx \tag{44}$$

$$E\left(X_R\left(D_a\right)\right)-E\left(X_R\left(D_b\right)\right)=E(\kappa)\left(\int_0^{D_b} S_X(x)dx-\int_0^{D_a} S_X(x)dx\right). \tag{45}$$

Dividing equation (45) through by $E(\kappa)$ and combine the integrals in the interval $[D_a, D_b]$, we have

$$\int_{D_a}^{D_b} S_X(x)dx=\frac{E\left(X_R\left(D_a\right)\right)-E\left(X_R\left(D_b\right)\right)}{E(\kappa)}. \tag{46}$$

The value of $X_R(D)$ at some arbitrary points of deductibility D can be expressed in terms of its value at a fixed point D_a and the derivatives of $X_R(D)$ is evaluated at that point D_a. Consequently, by expanding in Taylor's series, we have

$$E\left(X_R(D)\right)-E\left(X_R\left(D_a\right)\right)=\left(D-D_a\right)E\left(X_R'\left(D_a\right)\right)+\frac{\left(D-D_a\right)^2}{2}E\left(X_R''\left(D_a\right)\right)+\ldots \tag{46a}$$

Substituting $D=D_b$ in (46a), we have and terminating after the first term on the right, we have

$$E\left(X_R\left(D_b\right)\right)-E\left(X_R\left(D_a\right)\right)=\left(D_b-D_a\right)E\left(X_R'\left(D_a\right)\right). \tag{46b}$$

Equation (46b) implies that we should multiply (46) by $\frac{1}{D_a-D_b}$ and take limits as $D_a \to D_b$

$$\lim_{D_a\to D_b}\left[\frac{1}{D_a-D_b}\int_{D_a}^{D_b} S_X(x)dx\right]=\lim_{D_a\to D_b}\left[\frac{E\left(X_R\left(D_a\right)\right)-E\left(X_R\left(D_b\right)\right)}{E(\kappa)\left(D_a-D_b\right)}\right]; \tag{47}$$

$$\lim_{D_a\to D_b}\left[\frac{1}{D_a-D_b}\int_{D_a}^{D_b} S_X(x)dx\right]=\frac{1}{E(\kappa)}\lim_{D_a\to D_b}\left[\frac{E\left(X_R\left(D_a\right)\right)-E\left(X_R\left(D_b\right)\right)}{\left(D_a-D_b\right)}\right]$$
$$=\frac{1}{E(\kappa)}\frac{d}{dD}E\left(X_R(D)\right) \tag{47a}$$

Based on the definitions in Bahnemann (2015), we prove the following theorem.

Theorem 2.3: If

$$R_{INSURER}(X,D)=\begin{cases} X & X\le D \\ D & D<X<\infty \end{cases}=\psi\ ; \tag{48}$$

$$R_{REINSURER}\left(X,D\right)=\begin{cases}0 & 0<X\leq D\\ X-D & D<X<\infty\end{cases}=\xi\,. \tag{49}$$

Suppose further that

$$f_X\left(x\right)=\alpha e^{-\alpha x}\quad \text{for } x>0, \tag{50}$$

with mean

$$E\left(X\right)=\frac{1}{\beta}\quad \text{for } x>0. \tag{51}$$

Then

$$E\left(R_{INSURER}\left(X,D\right)\right)=\frac{1}{\beta}-e^{(-\alpha D)}E\left(\psi\right). \tag{52}$$

Proof:

$$\Pr\left(X>D\right)=S_X\left(D\right)=e^{-\alpha D}; \tag{53}$$

$$\Pr\left(X>D\right)=S_X\left(D\right)=e^{-D\frac{1}{\beta}}; \tag{54}$$

$$E\left(R_{INSURER}\left(X,D\right)\right)=\int_0^D xf_X\left(x\right)dx+\int_D^\infty xf_X\left(x\right)dx; \tag{55}$$

$$E\left(R_{INSURER}\left(X,D\right)\right)=\int_0^\infty xf_X\left(x\right)dx-\int_D^\infty xf_X\left(x\right)dx+\int_D^\infty xf_X\left(x\right)dx; \tag{56}$$

$$E\left(R_{INSURER}\left(X,D\right)\right)=E\left(x\right)-\int_D^\infty\left(x-D\right)f_X\left(x\right)dx. \tag{57}$$

Let

$$\psi=x-D\Rightarrow x=\psi+D; \tag{58}$$

$$dx=d\psi; \tag{59}$$

$$E\left(R_{INSURER}\left(X,D\right)\right)=E\left(x\right)-\int_0^\infty\psi\, f_X\left(\psi+D\right)d\psi; \tag{60}$$

$$f_X\left(x\right)=\alpha e^{-\alpha x}; \tag{61}$$

$$f_X\left(\psi+D\right)=\alpha e^{-\alpha(\psi+D)}; \tag{62}$$

$$E\left(R_{INSURER}\left(X,D\right)\right)=\frac{1}{\beta}-\int_0^\infty\psi\alpha e^{-\alpha(\psi+D)}d\psi; \tag{63}$$

$$E\left(R_{INSURER}\left(X,D\right)\right)=\frac{1}{\beta}-\int_0^\infty\psi\alpha e^{(-\alpha\psi)}e^{(-\alpha D)}d\psi; \tag{64}$$

$$E\left(R_{INSURER}\left(X,D\right)\right)=\frac{1}{\beta}-e^{(-\alpha D)}\int_{0}^{\infty}\psi\alpha e^{(-\alpha\psi)}d\psi\ ; \qquad (65)$$

But $$\int_{0}^{\infty}\psi\alpha e^{(-\alpha\psi)}d\psi=E\left(\psi\right) \qquad (66)$$

Hence $$E\left(R_{INSURER}\left(X,D\right)\right)=\frac{1}{\beta}-e^{(-\alpha D)}E\left(\psi\right); \qquad (67)$$

$$E\left(R_{REINSURER}\left(X,D\right)\right)=E\left(\zeta\right)=\int_{0}^{\infty}xdF_X\left(x+D\right)=\int_{D}^{\infty}\left(\zeta-D\right)dF_X\left(\zeta\right); \qquad (68)$$

$$E\left(R_{REINSURER}\left(X,D\right)\right)=\int_{0}^{\infty}\zeta dF_X\left(\zeta\right)-\int_{0}^{D}\zeta dF_X\left(\zeta\right)-D\int_{D}^{\infty}f_X\left(\zeta\right)d\zeta\ ; \qquad (68a)$$

$$E\left(R_{REINSURER}\left(X,D\right)\right)=\int_{0}^{\infty}\zeta f_X\left(\zeta\right)d\zeta-\int_{0}^{D}\zeta f_X\left(\zeta\right)d\zeta-D\int_{D}^{\infty}f_X\left(\zeta\right)d\zeta\ ; \qquad (68b)$$

$$E\left(R_{REINSURER}\left(X,D\right)\right)=Exp\left(\mu+\frac{1}{2}\sigma^2\right)-\left(Exp\left(\mu+\frac{\sigma^2}{2}\right)\Phi\left(\frac{\log_e D-\mu-\sigma^2}{\sigma}\right)\right)-D\int_{D}^{\infty}f_X\left(\zeta\right)d\zeta \qquad (68c)$$

Theorem 2.4: Let

$$E\left(R_{INSURER}\left(X,D\right)\right)=E\left(x\right)-E\left(x-D\right)^{+}, \qquad (69)$$

and if $X \sim Lognormal\left(\mu,\sigma^2\right)$, then

$$E\left(R_{INSURER}\left(X,D\right)\right)=Exp\left(\mu+\frac{1}{2}\sigma^2\right)\left\langle 1-\left\{\left[\left(1-\sigma\right)-\left(\frac{\log_e D-\mu}{\sigma}\right)\right]\right\}\right\rangle+D\Pr\left[Z>\frac{\log_e x-\mu}{\sigma}\right] \qquad (70)$$

Proof: We recognize that

$$E\left(x-D\right)^{+}=\int_{D}^{\infty}\left(x-D\right)f_X\left(x\right)dx\,. \qquad (70a)$$

Hence

$$E\left(R_{INSURER}\left(X,D\right)\right)=E\left(x\right)-\int_{D}^{\infty}\left(x-D\right)f_X\left(x\right)dx\,. \qquad (70b)$$

For the log-normally distributed claim, we have that

$$f_X\left(x\right)=\frac{1}{\sigma\sqrt{2\pi}}\frac{1}{x}e^{-\frac{\left[\left(\log_e x-\mu\right)^2\right]}{2\sigma^2}}\ ; \qquad (71)$$

$$\int_0^D \psi f_X(\psi)\,d\psi = Exp\left(\mu+\frac{\sigma^2}{2}\right)\Phi\left(\frac{\log_e D-\mu-\sigma^2}{\sigma}\right). \tag{72}$$

Observe that the first term in (72)

$$E(x) = Exp\left(\mu+\frac{1}{2}\sigma^2\right); \tag{73}$$

$$E\left(R_{INSURER}(X,D)\right) = E(x) - \int_D^\infty (x-D)\,f_X(x)\,dx = E(x) - \left[\int_D^\infty x f_X(x)\,dx - DS_X(x)\right]; \tag{74}$$

$$E\left(R_{INSURER}(X,D)\right) = E(x) - \int_D^\infty (x-D)\,f_X(x)\,dx = E(x) - \int_D^\infty x f_X(x)\,dx + DS_X(x); \tag{75}$$

$$\int_D^\infty x f_X(x)\,dx = \int_D^\infty x\frac{1}{\sigma\sqrt{2\pi}}\frac{1}{x}e^{-\frac{\left[(\log_e x-\mu)^2\right]}{2\sigma^2}}dx; \tag{76}$$

$$\int_D^\infty x f_X(x)\,dx = \int_D^\infty \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left[(\log_e x-\mu)^2\right]}{2\sigma^2}}dx; \tag{77}$$

Following Tse (2009),

$$\int_D^\infty x f_X(x)\,dx = E(x)\left[1-\Phi(\bar{z})\right]; \tag{78}$$

where

$$(1-\sigma)-\left(\frac{\log_e D-\mu}{\sigma}\right) = \Phi(\bar{z}); \tag{78a}$$

$$\int_D^\infty x f_X(x)\,dx = Exp\left(\mu+\frac{1}{2}\sigma^2\right)\left[(1-\sigma)-\left(\frac{\log_e D-\mu}{\sigma}\right)\right]; \tag{79}$$

$$E\left(R_{INSURER}(X,D)\right) = E(x) - \int_D^\infty x f_X(x)\,dx + DS_X(x); \tag{80}$$

$$E\left(R_{INSURER}(X,D)\right) = E(x) - \left\{Exp\left(\mu+\frac{1}{2}\sigma^2\right)\left[(1-\sigma)-\left(\frac{\log_e D-\mu}{\sigma}\right)\right]\right\} + DS_X(x) \tag{81}$$

$$E\left(R_{INSURER}(X,D)\right) = Exp\left(\mu+\frac{1}{2}\sigma^2\right) - \left\{Exp\left(\mu+\frac{1}{2}\sigma^2\right)\left[(1-\sigma)-\left(\frac{\log_e D-\mu}{\sigma}\right)\right]\right\} + DS_X(x) \tag{82}$$

Consequently, the insurance loss has been minimized by the value

$$\text{Re}\,duction = \left\{ Exp\left(\mu + \frac{1}{2}\sigma^2 \right) \left[(1-\sigma) - \left(\frac{\log_e D - \mu}{\sigma} \right) \right] \right\}; \tag{83}$$

$$E\left(R_{INSURER}(X,D)\right) = Exp\left(\mu + \frac{1}{2}\sigma^2 \right) \left\langle 1 - \left\{ \left[(1-\sigma) - \left(\log_e D^{\frac{1}{\sigma}} - \frac{\mu}{\sigma} \right) \right] \right\} \right\rangle + DS_X(x) \tag{84}$$

Observe that $S_X(D) = \Pr\left[Z > \frac{\log_e x - \mu}{\sigma} \right]$ **and substituting this in** (83) above, we have

$$E\left(R_{INSURER}(X,D)\right) = Exp\left(\mu + \frac{1}{2}\sigma^2 \right) \left\langle 1 - \left\{ \left[(1-\sigma) - \left(\log_e D^{\frac{1}{\sigma}} - \frac{\mu}{\sigma} \right) \right] \right\} \right\rangle + D \Pr\left[Z > \frac{\log_e x - \mu}{\sigma} \right] \tag{85}$$

3.0 MATERIAL AND METHODS

In this study, we shall apply both lognormal and exponential probability densities to enable us compute and compare cost per-loss, cost per-payment, loss elimination ratio under exponentially and Log-normally distributed severities for policies with deductible clauses. The appropriate properties of the two densities which will assist us in our computations will be discussed. Usually, loss modeling process is carried out through the use of appropriate continuous probability distributions whose expectations describe the severity value that scheme holders could claim. A random variable Y is said to be log-normally distributed if its probability density function is given by

$$g_Y(z) = \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{z} e^{-\frac{\left[(\log_e z - \mu)^2\right]}{2\sigma^2}}. \tag{85a}$$

Loss distribution function of lognormal distribution is given by

$$G_Y(z) = \frac{1}{\sqrt{2\pi\sigma}} \int_D^y \frac{1}{z} e^{-\frac{\left[(\log_e z - \mu)^2\right]}{2\sigma^2}} dz, \tag{86}$$

and the moments are defined as

$$E\left(Y^r\right) = e^{\left(r\mu + \frac{1}{2}r^2\sigma^2\right)},\ E(Y) = e^{\left(\mu + \frac{1}{2}\sigma^2\right)},\ E\left(Y^2\right) = e^{\left(2\mu + 2\sigma^2\right)}. \tag{87}$$

The variance of the risk is

$$Var(y) = e^{\left(2\mu + 2\sigma^2\right)} - \left[e^{\left(\mu + \frac{1}{2}\sigma^2\right)} \right]^2. \tag{88}$$

For both standard parametric estimation methods, the estimators are obtained in closed form. The method of moment estimations is defined as follows.

$$\hat{\mu} = \log_e\left(\frac{1}{k}\sum_{j=1}^{k} x_j\right)^2 - \log_e\sqrt{\left(\frac{1}{k}\sum_{j=1}^{k} x_j^2\right)} = \log_e\left(\frac{\left(\frac{1}{k}\sum_{j=1}^{k} x_j\right)^2}{\sqrt{\left(\frac{1}{k}\sum_{j=1}^{k} x_j^2\right)}}\right); \quad (89)$$

$$\hat{\sigma}^2 = \log_e\left(\frac{1}{k}\sum_{j=1}^{k} x_j^2\right)^2 - \log_e\left(\frac{1}{k}\sum_{j=1}^{k} x_j\right)^2 = \log_e\left(\frac{\left(\frac{1}{k}\sum_{j=1}^{k} x_j^2\right)^2}{\left(\frac{1}{k}\sum_{j=1}^{k} x_j\right)^2}\right). \quad (90)$$

The log-normal distribution is preferred when modelling claim sizes because it is skewed to the right and has a thick tail. When $\sigma < \in$ is a small number, then it bears semblance with normal distributions and this condition may be not be feasible. It is infinitely divisible and closed under scale and power transformations. Nevertheless, the Laplace transform has no definite closed form mathematical representations and the moment generating function is non-existing.

We observe that μ and σ^2 do not necessarily describe the mean and variance of y but are somewhat logarithmic in form. However, in Bahnemann et. al. (2015), the density of an exponentially distributed random variable is given by

$$g(x) = \alpha e^{-\alpha x} I_{[0,\infty)}(x); \quad (91)$$

$$G(x) = 1 - e^{-\alpha x} \quad \text{for } x > 0. \quad (92)$$

The Laplace transform is given as

$$L(s) = \int_0^\infty e^{-sx} g(x)\,dx = \frac{\alpha}{s+\alpha} \quad \text{for } s > -\alpha; \quad (93)$$

$$m_r = (-1)^r \left.\frac{\partial^r L(s)}{\partial s^r}\right|_{s=0} = \frac{r!}{\alpha^r} \quad \text{for } x > 0; \quad (94)$$

$$E(x) = \frac{1}{\alpha} \text{ and } Var(x) = \frac{1}{\alpha^2}. \quad (95)$$

The r^{th} root of Laplace transform is given as

$$L(s) = \left(\frac{\alpha}{s+\alpha}\right)^{\frac{1}{r}}, \quad (96)$$

making it to be divisible. Despite the fact that its density function exponentially decays, it is useful in developing insurance risk models. The expected loss under log-normally distributed risk with deductible conditions is defined by

$$\langle Y_L \rangle = \int_D^\infty (y-D) g_Y(y) dy = \int_D^\infty y g_Y(y) dy - \int_D^\infty D g_Y(y) dy ; \quad (97)$$

$$\langle Y_L \rangle = \int_D^\infty y g_Y(y) dy - D S_Y(D), \quad (98)$$

where

$$\int_D^\infty y g_Y(y) dy = \int_D^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\left[(\log_e y-\mu)^2\right]}{2\sigma^2}} dy \cong e^{\left[\frac{\sigma^2}{2}+\mu\right]\left\{1-\Phi\left(\frac{\ln D-\mu-\sigma^2}{\sigma}\right)\right\}} \quad (99)$$

$$S_Y(D) = \Pr\left[Z > \frac{\log_e y-\mu}{\sigma}\right]. \quad (100)$$

From the definition of log-normal distribution, we observe that

$$g_Y(y) = \frac{d}{dy}\Phi\left(\frac{(\log_e y-\mu)}{\sigma}\right) = \frac{1}{\sigma}\times\frac{1}{y}\times\Phi'\left(\frac{(\log_e y-\mu)}{\sigma}\right) \quad (101)$$

$$\Phi(y) = \int_0^y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds ; \Phi\left(\frac{(\log_e y-\mu)}{\sigma}\right) = \int_0^{\frac{(\log_e y-\mu)}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds . \quad (102)$$

Applying equation (20a) to differentiate equation (102), we obtain

$$g_Y(y) = \left(e^{-\frac{\left[(\log_e y-\mu)^2\right]}{2\sigma^2}}\right)\left(\frac{\sqrt{2\pi}}{y\times\sigma\times 2\pi}\right) \quad (103)$$

$y > 0$, location μ and scale $\sigma > 0$ and therefore,

$$\int_{-\infty}^y g_S(s) ds = \frac{(\log_e y-\mu)}{\sigma} . \quad (104)$$

The import of the paper lies in investigating techniques applicable in evaluating probability-based loss distributions as used in general insurance claims analysis. We see in Schlesinger (1985); Therese (2016), Liu and Wang (2017), and Woodard and Yi (2018) that under deductible conditions, the design of an adequate loss distribution which will model the severity of claims would enable insurance claim managers to have a good knowledge of claims data.

3.1 Data Presentation and Analysis: Mean Severity Under Exponential and Log-normal Distributions

In non-life insurance business, data on deductibles is often difficult to access because they represent personal retentions which are only borne by policy holders. We *extend* and further the scheme in Ogungbenle (2021a) by considering the rate relativity on deductible in order to compare the following standard empirical distributions by investigating the insured risk Y with unit sum insured under insurance cover and with specified deductibles D under exponential and lognormal distributions $0.1 \le D \le 1, Y \sim EXP(\alpha), \alpha = 1$ and severity when log-normally distributed as

$$Y \sim LN(\mu, \sigma^2), assume, \mu = -\frac{1}{2}, \sigma^2 = 1$$

3.2 Exponential distribution

$$S_Y(D) = e^{-\alpha D}, g_Z(D) = \alpha e^{-\alpha D}, H_Y(y) = \frac{g_Y(D)}{S_Y(D)} \tag{105}$$

$$E(Y_R(0.15)) = \langle (y-0.15)_+ \rangle = \int_{0.15}^{\infty} e^{-y} dy = e^{-0.15} = 0.86071; \tag{106}$$

$$S_Y(D) = e^{-0.15} = 0.86071 \Rightarrow \frac{\langle Y_L \rangle}{S_Y(D)} = \langle Y_P \rangle_{\exp} = \int_0^{\infty} e^{-y} dy = 1. \tag{107}$$

$\langle Y_p \rangle = \int_0^{\infty} g_Y(y) dy = 1$, hence, we can see that $E(Y_R(D)) < \langle Y_p \rangle$ that is the cost per loss amount is less than the cost per payment amount.

Table 1: Computed values of cost per loss and loss elimination ratio for exponentially distributed claim Source: Ogungbenle (2021a)

RELATIVITY DOMAIN	COST PER LOSS	LOSS RATIO	LOSS ELIMINATION RATIO-LER	CHANGE IN LER	COST PER PAYMENT
0.1	0.904837	0.095163	0.095163	0.0952	1
0.15	0.860708	0.139292	0.139292	0.0441	1
0.2	0.818731	0.181269	0.181269	0.042	1
0.25	0.778801	0.221199	0.221199	0.0399	1
0.3	0.740818	0.259182	0.259182	0.038	1
0.35	0.704688	0.295312	0.295312	0.0361	1
0.4	0.67032	0.32968	0.32968	0.0344	1
0.45	0.637628	0.362372	0.362372	0.0327	1
0.5	0.606531	0.393469	0.393469	0.0311	1
0.55	0.57695	0.42305	0.42305	0.0296	1
0.6	0.548812	0.451188	0.451188	0.0281	1
0.65	0.522046	0.477954	0.477954	0.0268	1
0.7	0.496585	0.503415	0.503415	0.0255	1
0.75	0.472367	0.527633	0.527633	0.0242	1
0.8	0.449329	0.550671	0.550671	0.023	1
0.85	0.427415	0.572585	0.572585	0.0219	1
0.9	0.40657	0.59343	0.59343	0.0208	1
0.95	0.386741	0.613259	0.613259	0.0198	1
1	0.367879	0.632121	0.632121	0.0189	1

The loss eliminated (LE) and loss elimination ratio (LER) are as given below:

$$X_L^P(D)=\left\{E\left(X_R(D)\right),\left\langle Y_p\right\rangle\right\}; \quad (108)$$

$$LE(y)=\frac{1}{\alpha}-\left\langle Y_L\right\rangle=1-\left\langle Y_L\right\rangle, LER(y)=\frac{\left\langle Y\right\rangle-E\left(Y_R(D)\right)}{\left\langle Y\right\rangle}. \quad (109)$$

3.3 Lognormal distribution

$$E\left(Y_R(D)\right)=\int_D^{\infty}(y-D)g_Y(y)dy=\int_D^{\infty}yg_Y(y)dy-\int_D^{\infty}Dg_Y(y)dy=$$

$$\int_D^{\infty}yg_Y(y)dy-DS_Y(D) \quad (110)$$

$$\int_D^{\infty}yg_Y(y)dy=\int_D^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left[(\log_e y-\mu)^2\right]}{2\sigma^2}}dy\cong e^{\left[\frac{\sigma^2}{2}+\mu\right]}\left\{1-\Phi\left(\frac{\ln D-\mu-\sigma^2}{\sigma}\right)\right\} \quad (111)$$

$$S_Y(D)=\Pr\left[Z>\frac{\log_e y-\mu}{\sigma}\right]. \quad (112)$$

Table 2: Computed values of log-normally distributed cost per loss and cost per payment losses

1	2	3	4	5	6	7	8	9	10
0.1	-2.3025851	-1.8025851	-1.8025851	-2.8025851	0.99746	0.00254	0.9642	0.90104	0.9344949
0.15	-1.89712	-1.39712	-1.39712	-2.39712	0.99176	0.00824	0.9188	0.85394	0.9294079
0.2	-1.6094379	-1.1094379	-1.1094379	-2.1094379	0.9825	0.0175	0.8661	0.80928	0.9343956
0.25	-1.3862944	-0.8862944	-0.8862944	-1.8862944	0.9703	0.0297	0.8122	0.76725	0.9446565
0.3	-1.2039728	-0.7039728	-0.7039728	-1.7039728	0.9558	0.0442	0.7592	0.72804	0.9589568
0.35	-1.0498221	-0.5498221	-0.5498221	-1.5498221	0.9393	0.0607	0.7085	0.691325	0.9757586
0.4	-0.9162907	-0.4162907	-0.4162907	-1.4162907	0.9215	0.0785	0.6613	0.65698	0.9934674
0.45	-0.7985077	-0.2985077	-0.2985077	-1.2985077	0.903	0.097	0.6172	0.62526	1.013059
0.5	-0.6931472	-0.1931472	-0.1931472	-1.1931472	0.8836	0.1164	0.5765	0.59535	1.0326973
0.55	-0.597837	-0.097837	-0.097837	-1.097837	0.864	0.136	0.5391	0.567495	1.0526711
0.6	-0.5108256	-0.0108256	-0.0108256	-1.0108256	0.844	0.156	0.5044	0.54136	1.0732752
0.65	-0.4307829	0.0692171	0.0692171	-0.9307829	0.8241	0.1759	0.4723	0.517105	1.0948656
0.7	-0.3566749	0.1433251	0.1433251	-0.8566749	0.8042	0.1958	0.4431	0.49403	1.1149402
0.75	-0.2876821	0.2123179	0.2123179	-0.7876821	0.7847	0.2153	0.416	0.4727	1.1362981
0.8	-0.2231436	0.2768564	0.2768564	-0.7231436	0.7651	0.2349	0.3909	0.45238	1.1572781
0.85	-0.1625189	0.3374811	0.3374811	-0.6625189	0.7464	0.2536	0.3681	0.433515	1.1777099
0.9	-0.1053605	0.3946395	0.3946395	-0.6053605	0.7273	0.2727	0.3464	0.41554	1.1995958
0.95	-0.0512933	0.4487067	0.4487067	-0.5512933	0.7091	0.2909	0.3268	0.39864	1.2198286
1	0	0.5	0.5	-0.5	0.6915	0.3085	0.3085	0.383	1.2414911

$$COLUMN\ 1=0.1\le D\le 1; COLUMN\ 2=\log_e D; COLUMN3\ =\log_e D-\mu;$$

$$COLUMN4\ =\frac{(\log_e D-\mu)}{\sigma} \quad (113)$$

$$COLUMN\ 5=\frac{(\log_e D-\mu)}{\sigma}-\sigma; COLUMN6\ =\int_D^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left[(\log_e y-\mu)^2\right]}{2\sigma^2}}dy. \quad (114)$$

$$COLUMN7 \ = \Phi\left(\frac{(\log_e D-\mu)}{\sigma}-\sigma\right); COLUMN8 \ = \Pr\left(Z>\frac{(\log_e D-\mu)}{\sigma}\right). \qquad (115)$$

The cost per loss under lognormal is computed in column 9 using equation (110), while the cost per payment under lognormal is computed in column 10 as given below.

$$COLUMN10 \ = E\left(Y_R\left(D\right)\right):\left\langle Y_P\right\rangle_{\log-normal} = \frac{E\left(Y_R\left(D\right)\right)}{S_Y\left(D\right)}. \qquad (116)$$

In order to see the trend of the change in the loss eliminated in the domain for D over which it has been defined, it was discovered that $E\left(Y_R\left(D\right)\right)<\left\langle Y_p\right\rangle$, that is the cost per loss amount is less than the cost per payment amount in either case of the two models.

4. RESULTS AND DISCUSSION

Following the result in Dupacova et al. (2003) while ignoring $DS_X\left(x\right)$, the term $E\left(x-D\right)^{+}$ at the right hand side of equation (69) can be expressed as follows:

$$E\left(x-D\right)^{+} = \int_D^{\infty}\left(x-D\right) f_X\left(x\right)dx = Exp\left(\mu+\frac{1}{2}\sigma^2\right)-\Phi\left(\frac{\mu-\log_e D+\sigma^2}{\sigma}\right)- D\Phi\left(\frac{\mu-\log_e D}{\sigma}\right) \qquad (117)$$

From our observation in Table 1, the maximum value of D is 1. Consequently, substituting $D=1$ in equation (117) above, we obtain,

$$E\left(x-1\right)^{+} = \int_1^{\infty}\left(x-1\right) f_X\left(x\right)dx = Exp\left(\mu+\frac{1}{2}\sigma^2\right)-\Phi\left(\frac{\mu-\log_e 1+\sigma^2}{\sigma}\right)- \Phi\left(\frac{\mu-\log_e 1}{\sigma}\right) \qquad (118)$$

Observe that $\log_e 1=0$, consequently, equation (118) becomes

$$E\left(x-1\right)^{+} = \int_1^{\infty}\left(x-1\right) f_X\left(x\right)dx = exp\left(\mu+\frac{1}{2}\sigma^2\right)-\left\{\Phi\left(\frac{\mu+\sigma^2}{\sigma}\right)+\Phi\left(\frac{\mu}{\sigma}\right)\right\} \qquad (119)$$

where $E\left(x\right)-E\left(x-1\right)^{+} = \Phi\left(\frac{\mu+\sigma^2}{\sigma}\right)+\Phi\left(\frac{\mu}{\sigma}\right)$ (120)

and $\Phi\left(.\right)$ is the distribution function for the standard normal distribution $N\left(0,1\right)$. In table 1 above, the loss elimination ratio shows the ratio of the decrease in the expected payment with an ordinary deductible to the expected payment without the deductible. It is instructive to note that $Y \wedge D = Y - (Y - D)_+$ and consequently, this will represent a decrease in the overall losses which could define savings or retention to the scheme holder under deductible settings. From the foregoing we quickly observe that the loss elimination ratio computes the expected savings resulting from the deductibles when expressed as a percentage of the loss, a condition of no deductible.

Using the last column of the table 1 and beyond a defined threshold, it becomes apparent that high deductibles may not permit a reasonable proportion of the eliminated loss due to the underwriting firm. The underwriter could adopt table 1 as a guide to ascertain if the deductible agreed to, by the insured and the insurer offers a reasonable percentage of the losses eliminated to the underwriter. We note that the cost per payment $|\langle Y_p \rangle| = 1$ is uniformly constant in the interval of definition irrespective of the value of the deductible. However, Table 2 is more challenging to obtain as a result of the framework of lognormal distribution but leads systematically to the calculation of cost per loss payment $E(Y_R(D))$ as shown in column 9. The rate relativity $0.1 \le D \le 1$ of D in column 1 is the basis for obtaining the cost per loss but since its values fall within 0 and 1, the logarithmic values are negatives except at $D = 1$. A simple calculation of cost per payment $\langle Y_p \rangle_{\log normal}$ from table 2 reveals a progressive increase from 0.934495 to 1.241491. Consequently, $|\langle Y_p \rangle| < 1, \mathit{for}, 0.1 \le D < 0.4$ $|\langle Y_p \rangle| > 1, \mathit{for}, 0.45 \le D < 1$ and therefore, the insurer experiences higher cost per payment than expected. The underwriter is therefore advised to apply deductible in the second *subdomain* so as to reduce the number and magnitude of nuisance claims advised and discourage moral hazards.

Comparing mean values in tables 1 and 2 above, it is observed that despite lognormal severity distribution usually has a thicker tail than the exponential distribution in literature, its mean loss $E(Y_R(D))$ column is observed to be correspondingly lower in value than the values of exponential mean loss, $E(Y_R(D))_{\log-normal} < E(Y_R(D))_{\exp onential}$ in the interval $0.1 \le D < 0.75$. It is amazing to observe that within $0.75 \le D \le 1.0$, $E(Y_R(D))_{\log-normal} > E(Y_R(D))_{\exp onential}$. This is because irrespective of the values of the deductibles, the cost per payment under exponential distribution is constant at unity. However, the cost per payment under lognormal distribution steadily increases far above unity as relativity correspondingly increases within the interval $0.1 < D \le 1.0$. We should be reminded that the accuracy of the result depends very much on rate relativity collected for the analysis and this has a considerable application in actuarial literature. We need to stress that the deductibles have a significant effect on the number of payments such that if the deductible enforced increases, then the number of payments per period reduces and in the corollary, when the deductibles reduces in value, then the number of payment per period increases too. Let $Y(i)$ assume the ground up losses. Furthermore, let $\sum_L$ represent the total number of losses. Assuming that a deductible has been imposed. Let $\pi = \Pr(Y > D)$ defines the probability that a loss will translate to a payment function. Now, define the indicator function as

$$I(i) = \begin{cases} 1 \quad if \quad the \quad ith \quad loss \quad occurs \quad and \quad results \quad in \quad a \quad payment \\ \qquad 0 \quad if \quad it \quad is \quad otherwise \end{cases}$$

Consequently, $I(i)$ is assumes a Bernoulli random variable and as such we have that

$$\Pr[I(i)=1] = \Pr(Y > D) = \pi \text{ but } \Pr[I(i)=0] = 1 - \Pr(Y > D) = 1 - \pi .$$

The probability generating function is given by

$$P^{I(i)}(u) = (\Pr(Y > D))u - u + 1 = u(\pi - 1) + 1 .$$

Suppose $\sum_P$ represents the total number of payments, then

$$\sum_P = I(1) + I(2) + I(3) + ... + I(\sum_L) = \sum_{k=1}^{\sum_L} I(k) .$$

and hence $\sum_P$ becomes a compound frequency model or the aggregate claim models. If the collection $\{I(1), I(2), I(3), ..., I(\sum_L)\}$ are mutually independent, then $\sum_P = I(1) + I(2) + I(3) + ... + I(\sum_L) = \sum_{k=1}^{\sum_L} I(k)$ defines compound distribution and $\sum_L$ serve as the primary distribution and a Bernoulli distribution. Consequently, $P^{\sum_P}(u) = P^{\sum_L}(1 + \pi(u-1))$.

CONCLUSION

In the course of the data computations, we carried out comparative analysis of claim obtained from the two models above. This method can be used to estimate aggregate claims as the deductible level increases for every scheme holder and such that the estimated claims could be compared with the hypothetical observed claims which can be arrived at by applying the hypothetical deductible value to the background losses. This study provides a mathematical characterization of the deductible relativity for cost per loss insurance and cost per payment deductibles. Our contribution to literature is deep rooted in the generalization of coverage modifications to empirical data and the applications of coverage modifications theory to loss variable with continuous probability distribution functions where empirical data was used to compute claim value under the effect of coverage modifications of deductible.

We have established a deep relationship between severity and deductible which enables us to use a well-known density to deal with average loss functions. The deductible is meant to improve the underlying models of the underwriting business observed as a complete transfer of risk from scheme holder to the underwriter by introducing this variant of partial risk-transfer. As a result of this partial risk-transfer mechanism, prime interest will be connected to the computation of the correct pricing as the premium for risk bearing. The application of deductible will usually expand the basic claim models but in comparison to complete risk transfer, there does not exist any form of equality between claims and indemnity. We should observe that if the kind of deductible to be used is known, the collection of claim number; claim value and aggregate claim will form the basis for constructing models for the number of indemnity payments; the indemnity value and aggregate indemnity. The contractual estimation of a deductible can be obtained by the indemnity function that establishes a correspondence between claim and indemnity such as shown in equations (1), (2) and (48). Nonetheless, under a proportional deductible structure, there is no significant variation between claim number and indemnity number, however, for non-proportional deductibles, it is only the claims that are actually greater than the agreed deductible value that are usually indemnified and consequently the number of indemnities in comparison to the number of claims will be reduced essentially in the many insurance business bearing small deductibles because to a large extent, claim will concentrate around the neighbourhood of small insurance losses.

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