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RESEARCH ARTICLE

# RECONSTRUCTION OF HELM GRAPH AND WEB GRAPH

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## ABSTRACT

The graph reconstruction conjecture asserts that every simple undirected graph with $n$ number of vertices (where $n \geq 3$) is uniquely determined up to isomorphism, by its collection of unlabeled form of vertex deleted sub graphs. If a vertex deleted sub graph of $G$ is given in unlabeled form, then it is called a "card" of $G$. The collection of all cards of $G$ is called the "deck" of $G$, and is denoted by $D(G)$. If we can determine the original graph $G$ from the deck of the graph $G$, we can say that it is reconstructible. In this paper we propose a new method to reconstruct the helm graph $(H_n)$ and the web graph $(W_n)$, by using the degree sequence of its collection of vertex deleted sub graphs. Helm graph is obtained by adjoining a pendant edge to each node of the cycle of the $n$-wheel graph. Web graph is obtained by joining pendant vertices to the vertices in the outer cycle of the helm graph to form a cycle and again adding pendant vertices to the new cycle. For both graphs we found decks and obtained the degree sequence of each cards. Finally, we reconstructed the original graph using the degree sequences.

**Keywords**: *Reconstruction, Helm graph, Web graph*

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## 1. INTRODUCTION

The reconstruction conjecture is an interesting idea in graph theory which has remained open for many years. It states that any graph with at least three vertices is determined up to isomorphism by their collection of unlabeled form of vertex deleted sub graphs. A vertex deleted subgraph of $G$ is called a "card" of $G$. The multiset of graphs that is obtained from deleting one vertex in every possible way from the graph $G$ is referred to as the "deck" of the graph $G$ [1]. A graph is re-constructible, if it is uniquely determined up to isomorphism by its deck. Equivalently, $G$ is not re-constructible if and only if there is a graph $H$ such that $D(G) = D(H)$ [2].

Any two graphs $G$ and $H$ which have the same vertex set $V$ are said to be hypomorphic if, for all $v \in V$, their vertex deleted subgraphs $G - v$ and $H - v$ are isomorphic [3]. In 1960, S.M Ulam proposed a new problem as a set theory problem in his book "A Collection of Mathematical Pro-

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blems". Consider two sets $E$ and $F$ with $m$ elements, and he defined a distance function $\mu$ for every distinct pair of points, with values of either $1$ or $2$. The distance function of a point for itself will be zero i.e. $\mu(p,p) = 0$. If, for every subset of $n-1$ point of $E$, there exists an isometric system of $m-1$ points of $F$. His problem was that, does $E$ and $F$ isomorphic, when the number of distinct subsets isomorphic to any given subset of $m-1$ points is the same in $E$ and in $F$ [4].

The well-known Kelly-Ulam conjecture can be stated as follows; for any two simple connected hypomorphic graphs, $G$ and $H$, with $n$ vertices, where $n \geq 3$ and for a bijective function

$$f: V(G) \rightarrow V(H),$$
$$v_i \in V(G) \rightarrow u_i \in V(H),$$

such that

$$G_i \equiv G - v_i \cong H_i \equiv H - u_i \quad , \forall\, i\,, i = 1,2,\dots,n, \text{ then}$$
$$G \cong H$$

If two graphs have this map we called those two graphs are hypomorphic [5].

**Theorem 1.1: Ulam Conjecture**

Let $G$ and $H$ be graphs with vertex set $V(G)$ such that $V(G) = v_1, v_2, \dots, v_n$; where $n \geq 3$. Given that, $G - v_i \cong H - u_i$, then, $G \cong H$ [6]. This conjecture states that hypomorphism of any two graphs implies their isomorphism.

The current version of this problem, known as the reconstruction conjecture, which is formulated by Frank Harary. Reconstructible graphs are the graphs which are obeying the Reconstruction conjecture. P. J. Kelly first proved that trees are reconstructible, and his proof is quite lengthy. Therefore, a short proof was given by Greenwell and Hemminger using a simple counting theorem. [6]

We say that two graphs $G$ and $H$ are hypomorphic if there exists a bijection $f: V(G) \rightarrow V(H)$ such that $G - x$ is isomorphic to $H - f(x)$ for each $x \in V(G)$. Such a mapping $f$ is called a hypomorphism of $G$ to $H$. Obviously, each isomorphism is a hypomorphism, but the converse is not true [7].

If there exist $u, v \in V(G)$ such that $\{u,v\} \neq \{x,y\}$ and $G - \{x,y\}$ is isomorphic to $G - \{u,v\}$, for an unordered pair of vertices $\{x,y\}$ of a graph $G$, then we called it a 'bad pair' [8]. Xuding Zhu conjectured that, if a graph $G$ has a vertex $v$ which is contained in at most three bad pairs, then all hypomorphs of $G$ are isomorphic to $G$ and hence $G$ is reconstrcutible [8].

In this paper, we find an alternative way to reconstruct the helm graph and the web graph using their degree sequences of the deck.

## 2. MATERIAL AND METHODS

First, we stated some important definitions we used to get the result of this work.

**Definition 2.1 (Card of a graph $G$):** Let $G$ be a simple graph with $V$ vertices and $E$ edges. A vertex deleted subgraph $G - x$ ($x \in V$) of $G$ is called a "card' of $G$ [2].

**Definition 2.2 (Deck of a graph $G$):** The collection of all cards of graph $G$ is called the "deck' of $G$, and it is denoted by $D(G)$ [2].

**Definition 2.3 (Reconstruction of a graph $G$):** A graph $G$ with $V$ vertex set is said to be reconstructible if it is determined up to isomorphism from the collection of all vertex deleted unlabeled subgraphs (Deck) [9].

**Definition 2.4 (Helm Graph):** The helm graph $H_n$ is obtained by adjoining a pendant edge at each node of the outer cycle for a $n$-wheel graph [10].

**Definition 2.5 (Web Graph):** A web graph $W_n$ is obtained by adjoining the pendant vertices of a helm graph $H_n$ to form a cycle and then adding pendant vertices to the new cycle [11].

In our work, we have found an alternative way to reconstruct the helm graph and the web graph. First, consider the helm graph. The helm graph $H_n$ consist $n$ number of pendent vertices and $n$ number of vertices in the outer cycle of the graph and finally one middle vertex and we called it as "$hub$".

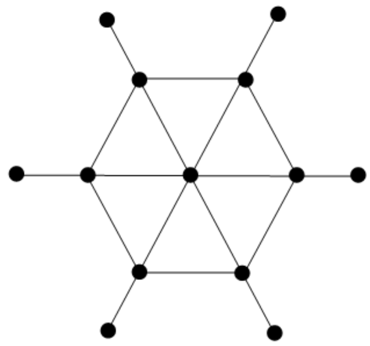

**Figure 1:** Helm graph

To find the deck of the helm graph, we have to delete one vertex and all the edges attached to it at each turn. Since helm graph $H_n$ has $2n + 1$ vertices, we will obtain $2n + 1$ number of cards in the deck.

Assume that the helm graph has $n$ number of pendant vertices. Figure 2 illustrates the helm graph with $(2n + 1)$ vertices. The graph has $(2n + 1)$ vertices and also the number of edges can be determined using

$$|E(G)| = \frac{\sum q_i}{p - 2}$$

where $q_i$ is the number of edges in $i^{th}$ deck and $p$ is the number of vertices in graph $G$.

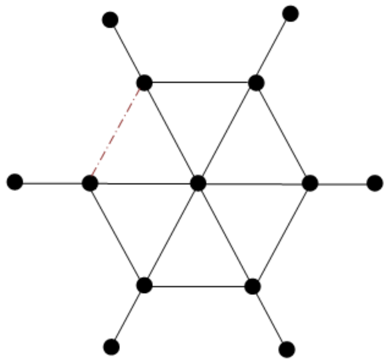

**Figure 2:** Helm graph $H_n$

Clearly, graph $H_n$ has $n$ vertices with degree $1$, and $n$ vertices with degree $4$, and $1$ vertex with degree $n$. Categorize these $(2n+1)$ vertices into three types according to their degree sequences. Let $U_i$ be the degree $1$ vertices, $V_i$ be the degree $4$ vertices, $i = 1,2, \dots, n$, and $W$ be the degree $n$ vertex.

By removing degree $1$ vertices $(U_i)$ from helm graph $H_n$ we obtained $n$ number of copies of the following graph as shown in Figure 3.

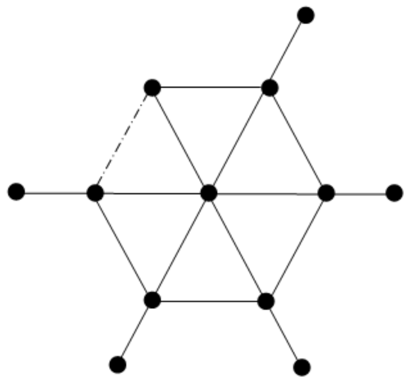

**Figure 3:** Degree $1$ vertex deleted sub graph of Helm graph $H_n$

Secondly, remove degree $4$ vertices $(\ V_i)$. Then, we obtained $n$ number of copies of the graph as shown in Figure 4.

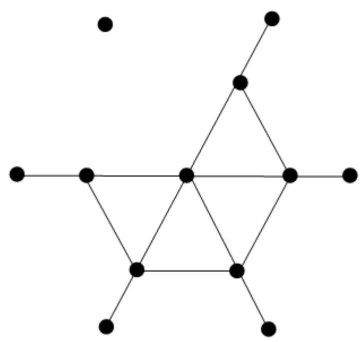

**Figure 4:** Degree $4$ vertex deleted subgraph of Helm graph $H_n$

Finally, remove degree $n$ vertex $W$ (or the middle vertex/hub). Then, we obtained the graph as shown in Figure 5.

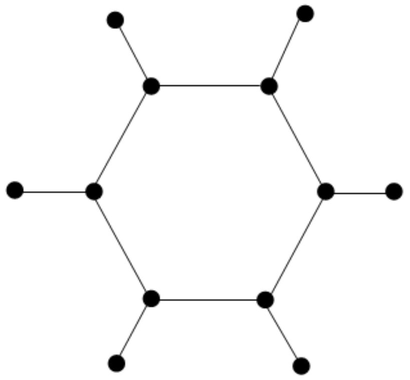

**Figure 5:** Degree $n$ vertex deleted subgraph of Helm graph $H_n$

We can observe that there are three types of cards. Find the degree sequence of each type and we get,

I. $C_1 = \{n, \underbrace{4, 4, \dots, 4}_{n-1}, 3, \underbrace{1, 1, \dots, 1}_{n-1}\}$

II. $C_2 = \{n-1, \underbrace{4, 4, \dots, 4}_{n-3}, 3, 3, \underbrace{1, 1, \dots, 1}_{n-1}, 0\}$

III. $C_3 = \{\underbrace{3, 3, \dots, 3}_{n}, \underbrace{1, 1, \dots, 1}_{n}\}$

Now consider the web graph. Web graph $W_n$ is obtained by joining pendant vertices to the vertices in the outer cycle of the helm graph $H_n$ to form a new cycle and again adding pendant vertices to the new cycle, see Figure 6.

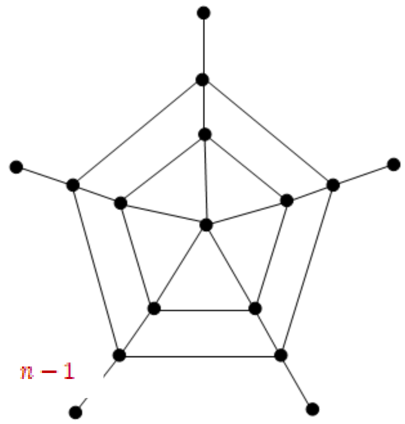


**Figure 6:** Web graph $W_5$

To find the deck of the web graph, we have to delete one vertex and all the edges attached to it at each turn. Since web graph $W_n$ has $3n+1$ vertices, we will obtain $3n+1$ number of cards in the deck.

Assume that the web graph has $n$ number of pendant vertices. Figure 7 illustrates the web graph with $(3n+1)$ vertices.

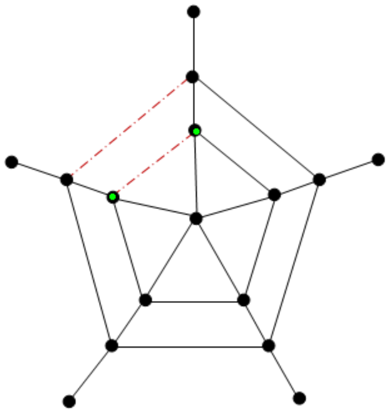

**Figure 7:** Web graph $W_n$

Similar to the helm graph, web graph also has $n$ vertices with degree $1$, and $1$ vertex with degree $n$. The only difference is web graph has $2n$ number of degree-$4$ vertices.

Divide these $(3n+1)$ vertices into three types according to their degree sequences. Let $P_i$ be the degree $1$ vertices, $Q_i$ be the degree $4$ vertices in outer cycle, $Q'_i$ be the degree $4$ vertices in inner cycle, where $i = 1,2,\dots,n$, and $R$ be the degree $n$ vertex.

First, remove degree $1$ vertices $(P_i)$ from helm graph $H_n$ and we obtained $n$ number of copies of the graph as shown in Figure 8.

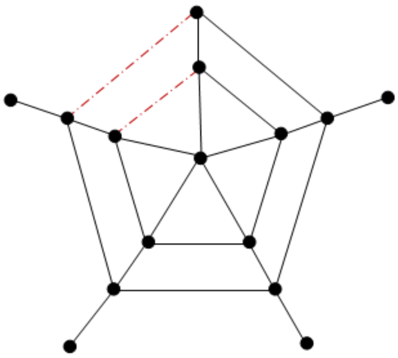

**Figure 8:** Degree $1$ vertex deleted sub graph of Web graph $W_n$

Secondly, remove degree $4$ vertices $(Q_i)$. Those vertices are on two cycles. Therefore, at first we remove the vertices in the outer cycle and then we remove the vertices in the inner cycle. After that, we obtained $2n$ number of copies of the graph as shown in Figure 9.

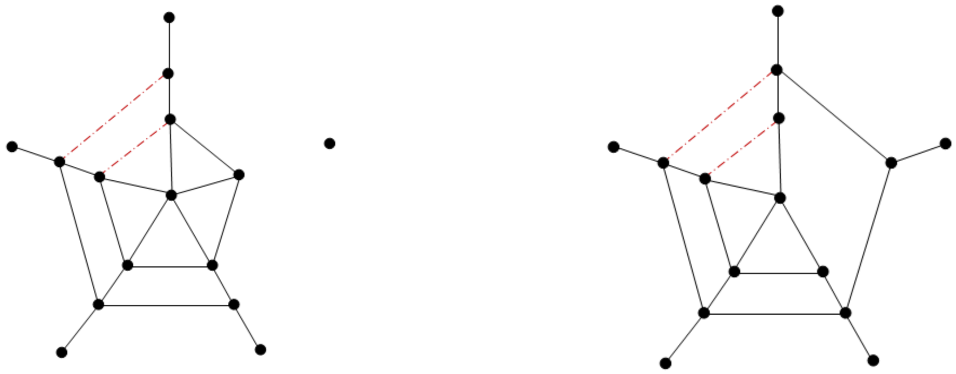

**Figure 9:** Two types of degree $1$ vertex deleted sub graph of Web graph $W_n$

Finally, remove degree $n$ vertex $R$ (or the middle vertex/hub). Then we obtained one of the following graphs.

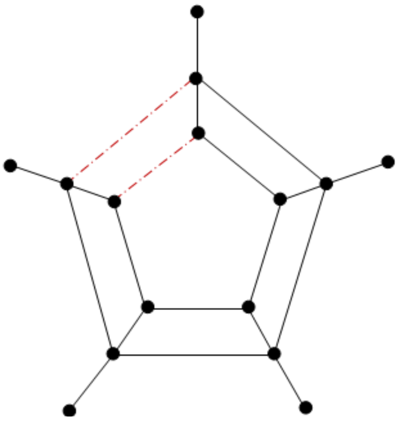

**Figure 10:** Degree $n$ vertex deleted sub graph of Web graph $W_n$

We can observe that there are three types of cards. We get the degree sequence of each type as follows:

I. $C_1' = \{n, \underbrace{4, 4, \dots, 4}_{2n-1}, 3, \underbrace{1, 1, \dots, 1}_{n-1}\}$

II. $C_2' = \{n, \underbrace{4, 4, \dots, 4}_{2n-4}, 3, 3, 3, \underbrace{1, 1, \dots, 1}_{n-1}, 0\}$

III. $C_3' = \{n-1, \underbrace{4, 4, \dots, 4}_{2n-4}, 3, 3, 3, \underbrace{1, 1, \dots, 1}_{n}\}$

IV. $C_4' = \{\underbrace{4, 4, \dots, 4}_{n}, \underbrace{3, 3, \dots, 3}_{n}, \underbrace{1, 1, \dots, 1}_{n}\}$

## 3. RESULTS AND DISCUSSION

**Theorem 2.1:** Helm graph $H_n$ can be uniquely reconstructed by using the degree sequences of its deck.

**Proof:** Let us consider the three types of cards of Helm graph. By looking at those cards we can observe that the degree $n$ vertices $(W)$ connect only with $V_i$'s. i.e. there are no edges between $W$ and $U_i$'s. Also, each $V_i$ 's are adjacent to $W$ and two of $V_i$'s and one $U_i$.

Since the degree sequence $C_2$ has only two degree-$3$ vertices and only $(n-3)$ degree-$4$ vertices, one of $V_i$ should be deleted from the original graph to obtain the $C_2$ type. Moreover, $U_i$'s adjacent to only one $V_i$.

Hence, the Helm graph $H_n$ is uniquely reconstructible.

**Theorem 2.2:** Web graph $W_n$ can be uniquely reconstructed by using the degree sequences of its deck.

**Proof:** Let us consider the three types of cards of Web graph. By looking at those cards we can observe that the degree-$n$ vertices $(R)$ are connected only with $Q_i'$'s. i.e. there are no edges between $R$ and $Q_i$'s. Also, each $Q_i$'s are adjacent to $P_i$ and two of $Q_i$'s and one $Q_i'$.

Similarly, each $Q_i'$'s are adjacent to $Q_i$ and two of $Q_i'$'s and one $R$.

Since the degree sequences $C_2$ and $C_3$ have only three degree-$3$ vertices and only $(2n-4)$ degree-4 vertices, one of the vertex from $Q_i$ or $Q_i'$ should be deleted from the original graph to obtain the $C_2$ or $C_3$ type.

Moreover, $P_i$ adjacent to only one $Q_i$.
Hence, the Web graph $W_n$ is uniquely reconstructible.

## CONCLUSION

In our work, we found an alternative method to reconstruct the Helm graph and the Web graph using the degree sequences of their deck. First, we categorize every card of helm graphs into three types. Then we found a general way to write degree sequences for each type. Using those three degree sequences we conjecture that the Helm graph could be uniquely reconstructed. Likewise, we tried to separate every card in the Web graph by its similarities, and we were able to separate every card in the deck of the web graph into four categories. Finally, by observing the degree sequence of each type we conclude that web graph is also uniquely reconstructible.

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